

Motion of a Vortex Filament on a Slanted Plane

Masashi AIKI

Abstract

We consider a nonlinear model equation, known as the Localized Induction Equation, describing the motion of a vortex filament immersed in an incompressible and inviscid fluid. We prove the unique solvability of an initial-boundary value problem describing the motion of a vortex filament on a slanted plane.

1 Introduction and Problem Setting

A vortex filament is a space curve on which the vorticity of the fluid is concentrated. Vortex filaments are used to model very thin vortex structures such as vortices that trail off airplane wings or propellers. In this paper, we prove the solvability of the following initial-boundary value problem which describes the motion of a vortex filament moving on a slanted plane.

$$(1.1) \quad \begin{cases} \mathbf{x}_t = \mathbf{x}_s \times \mathbf{x}_{ss}, & s \in I, t > 0, \\ \mathbf{x}(s, 0) = \mathbf{x}_0(s), & s \in I, t > 0, \\ \mathbf{x}_s(0, t) = \mathbf{a}, \mathbf{x}_s(1, t) = \mathbf{e}_3, & t > 0, \end{cases}$$

where $\mathbf{x}(s, t) = {}^t(x_1(s, t), x_2(s, t), x_3(s, t))$ is the position vector of the vortex filament parametrized by its arc length s at time t , \times is the exterior product in the three dimensional Euclidean space, $I = (0, 1) \subset \mathbf{R}$ is an open interval, $\mathbf{a} \in \mathbf{R}^3$ is an arbitrary vector satisfying $|\mathbf{a}| = 1$, $\mathbf{e}_3 = {}^t(0, 0, 1)$, and subscripts s and t are differentiations with the respective variables. Problem (1.1) describes the motion of a segment of a vortex filament moving on a slanted plane. We can see that, by taking the trace $s = 0$ in the equation of problem (1.1), a filament moving according to problem (1.1) satisfies

$$\mathbf{x}_t(0, t) = \mathbf{a} \times \mathbf{x}_{ss}(0, t),$$

hence the end-point $\mathbf{x}(0, t)$ of the filament moves along the plane perpendicular to \mathbf{a} . The reason we also impose a boundary condition at $s = 1$ is for the following reason. A more intuitive problem setting for a vortex filament moving on a plane would be

$$(1.2) \quad \begin{cases} \mathbf{x}_t = \mathbf{x}_s \times \mathbf{x}_{ss}, & s > 0, t > 0, \\ \mathbf{x}(s, 0) = \mathbf{x}_0(s), & s > 0, t > 0, \\ \mathbf{x}_s(0, t) = \mathbf{a}, & t > 0, \end{cases}$$

which is a problem describing an infinitely long filament with one end moving along the plane perpendicular to \mathbf{a} . The solvability of problem (1.2) is a direct consequence of a previous work by the author and Iguchi [1], which proved the solvability of problem (1.2) with $\mathbf{a} = \mathbf{e}_3$, because the solution of problem (1.2) can be obtained by rotating the solution obtained in [1] in a way that \mathbf{a} is trasformed to \mathbf{e}_3 . Hence, problem (1.2) for general \mathbf{a} is essentially the same as the case $\mathbf{a} = \mathbf{e}_3$. So to describe the motion of a vortex filament on a slanted plane, we imposed a boundary condition at $s = 1$ to set a reference plane which allows us to express the slanted-ness of the plane that the filament is moving on. The motivation for considering problem (1.1) comes from the following problem.

$$(1.3) \quad \begin{cases} \mathbf{x}_t = \mathbf{x}_s \times \mathbf{x}_{ss}, & s > 0, t > 0, \\ \mathbf{x}(s, 0) = \mathbf{x}_0(s), & s > 0, \\ \mathbf{x}_s(0, t) = \frac{\nabla B(\mathbf{x}(0, t))}{|\nabla B(\mathbf{x}(0, t))|}, & t > 0, \end{cases}$$

where $\nabla = {}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and for ${}^t(x_1, x_2, x_3) \in \mathbf{R}^3$, $B : \mathbf{R}^3 \rightarrow \mathbf{R}$ is a given function of the form

$$B(x_1, x_2, x_3) = x_3 - b(x_2, x_3)$$

for a given scalar function $b : \mathbf{R}^2 \rightarrow \mathbf{R}$. Problem (1.3) describes an infinitely long vortex filament moving on a surface given as the graph of b in the three-dimensional Euclidean space. Problem (1.3) is a generalization of the problem setting in [1] and can be seen as a simplified model for the motion of a tornado, where the ground is given by the graph of b , but the solvability for a general b seems hard, and as a first step, we chose the special case where the ground is a slanted plane.

The equation in problem (1.1) is called the Localized Induction Equation (LIE) which is derived by applying the so-called localized induction approximation to the Biot–Savart integral. The LIE was first derived by Da Rios in 1906 and was re-derived twice independently by Murakami et al. in 1937 and by Arms and Hama in 1965. Many researches have been done on the LIE and many results have been obtained. Nishiyama and Tani [9, 10] proved the unique solvability of the initial value problem in Sobolev spaces. Koiso considered a geometrically generalized setting in which he proved rigorously the equivalence of the LIE and a nonlinear Schrödinger equation. This equivalence was first shown by Hasimoto [5] in which he studied the formation of solitons on a vortex filament. He defined a transformation of variable known as the Hasimoto transformation to transform the LIE into a nonlinear Schrödinger equation. The Hasimoto transformation was proposed by Hasimoto [5] and is a change of variable given by

$$\psi = \kappa \exp \left(i \int_0^s \tau \, ds \right),$$

where κ is the curvature and τ is the torsion of the filament. Defined as such, it is well known that ψ satisfies the nonlinear Schrödinger equation given by

$$(1.4) \quad i \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2} |\psi|^2 \psi.$$

The original transformation proposed by Hasimoto uses the torsion of the filament in its definition, which means that the transformation is undefined at points where the curvature of the filament is zero. Koiso [7] constructed a transformation, sometimes referred to as the generalized Hasimoto transformation, and gave a mathematically rigorous proof of the equivalence of the LIE and (1.4). More recently, Banica and Vega [2, 3] and Gutiérrez, Rivas, and Vega [4] constructed and analyzed a family of self-similar solutions of the LIE which forms a corner in finite time. The authors [1] proved the unique solvability of an initial-boundary value problem for the LIE in which the filament moved in the three-dimensional half space. Nishiyama and Tani [9] also considered initial-boundary value problems with different boundary conditions.

Setting $\mathbf{v} := \mathbf{x}_s$ and taking the s -derivative of the equation in problem (1.1), we see that problem (1.1) is transformed to

$$(1.5) \quad \begin{cases} \mathbf{v}_t = \mathbf{v} \times \mathbf{v}_{ss}, & s \in I, t > 0, \\ \mathbf{v}(s, 0) = \mathbf{v}_0(s), & s \in I, t > 0, \\ \mathbf{v}(0, t) = \mathbf{a}, \mathbf{v}(1, t) = \mathbf{e}_3, & t > 0, \end{cases}$$

where $\mathbf{v}_0 := \mathbf{x}_{0s}$. If we can prove the solvability of problem (1.5), the solution \mathbf{x} of problem (1.1) can be constructed from \mathbf{v} by

$$\mathbf{x}(s, t) = \mathbf{x}_0(s) + \int_0^t \mathbf{v}(s, \tau) \times \mathbf{v}_{ss}(s, \tau) d\tau,$$

hence the solvability of problem (1.1) and (1.5) are equivalent. To this end, we consider the solvability of problem (1.5) from here on.

The contents of the rest of the paper are as follows. In Section 2, we define notations used in this paper and state our main theorem. In Section 3, we introduce a regularized nonlinear problem and construct a corrected initial datum associated to the regularized problem. The correction is necessary to insure that the compatibility conditions for the regularized problem are satisfied. In Section 4, we give a brief description of the method used to prove the solvability of the linear problem associated to the regularized nonlinear problem given in Section 3 and state the existence theorem for the regularized nonlinear problem. Finally, in Section 5 we construct the solution of (1.5), and derive estimates of the solution to prove the time-global solvability of (1.5).

2 Function Spaces, Notations, and Main Theorem

We introduce some function spaces that will be used throughout this paper, and notations associated with the spaces. For a non-negative integer m , and $1 \leq p \leq \infty$, $W^{m,p}(I)$ is the Sobolev space containing all real-valued functions that have derivatives in the sense of distribution up to order m belonging to $L^p(I)$. We set $H^m(I) := W^{m,2}(I)$ as the Sobolev space equipped with the usual inner product. The norm in $H^m(I)$ is denoted by $\|\cdot\|_m$ and we simply write $\|\cdot\|$ for $\|\cdot\|_0$. Otherwise, for a Banach space X , the norm in X is written as $\|\cdot\|_X$. The inner product in $L^2(I)$ is denoted by (\cdot, \cdot) .

For $0 < T \leq \infty$ and a Banach space X , $C^m([0, T]; X)$ ($C^m([0, \infty); X)$ when $T = \infty$), denotes the space of functions that are m times continuously differentiable in t with respect to the norm of X .

For any function space described above, we say that a vector valued function belongs to the function space if each of its components does.

We further define notations to express the compatibility conditions for problem (1.5). First we set $\mathbf{P}_0(\mathbf{v}) := \mathbf{v}$ and $\mathbf{P}_1(\mathbf{v}) := \mathbf{v} \times \mathbf{v}_{ss}$. $\mathbf{P}_1(\mathbf{v})$ is the right-hand side of the equation in (1.5). We also use the notation $\mathbf{P}_1(s, t) := \mathbf{P}_1(\mathbf{v})$ and sometimes omit (s, t) for simplicity. We successively define

$$\mathbf{P}_m := \sum_{j=0}^{m-1} \binom{m-1}{j} \mathbf{P}_j \times \partial_s^2 \mathbf{P}_{m-1-j},$$

where $\binom{m-1}{j}$ is the binomial coefficient. \mathbf{P}_m gives the expression of $\partial_t^m \mathbf{v}$ with only s -derivatives of \mathbf{v} . From the boundary condition for problem (1.5), we arrive at the following definitions for the compatibility conditions.

Definition 2.1 (*Compatibility conditions for (1.5)*). For $m \in \mathbf{N} \cup \{0\}$, we say that \mathbf{v}_0 satisfies the m -th order compatibility condition for (1.5) if $\mathbf{v}_0 \in H^{2m+1}(I)$ and

$$\mathbf{v}_0(0) = \mathbf{a}, \quad \mathbf{v}_0(1) = \mathbf{e}_3,$$

when $m = 0$, and

$$\mathbf{P}_m(\mathbf{v}_0(0)) = \mathbf{P}_m(\mathbf{v}_0(1)) = \mathbf{0}$$

when $m \geq 1$. We also say that \mathbf{v}_0 satisfies the compatibility conditions for (1.5) up to order m if \mathbf{v}_0 satisfies the k -th order compatibility condition for all k with $0 \leq k \leq m$.

Now we state our main theorem regarding the solvability of (1.5).

Theorem 2.2 For an integer $l \geq 0$, if $\mathbf{v}_0 \in H^{l+3}(I)$, $|\mathbf{v}_0| \equiv 1$, and \mathbf{v}_0 satisfy the compatibility conditions up to order $[\frac{l+3}{2}]$, then there exists a unique solution \mathbf{v} satisfying $|\mathbf{v}| \equiv 1$ and

$$\mathbf{v} \in \bigcap_{j=0}^{[\frac{l+3}{2}]} C^j([0, \infty); H^{l+3-2j}(I)).$$

Here, $[\frac{l+3}{2}]$ is the largest integer not exceeding $\frac{l+3}{2}$.

The above theorem gives the time-global unique solvability of problem (1.5) and thus, for (1.1). Note that if the initial datum \mathbf{v}_0 satisfies $|\mathbf{v}_0| \equiv 1$, then a solution \mathbf{v} of (1.5) also satisfies $|\mathbf{v}| \equiv 1$ automatically. This is because

$$\frac{d}{dt} |\mathbf{v}|^2 = 2\mathbf{v} \cdot \mathbf{v}_t = 2\mathbf{v} \cdot (\mathbf{v} \times \mathbf{v}_{ss}) = 0,$$

and the arc length parameter is preserved throughout the motion. Here, \cdot is the inner product in the three-dimensional Euclidean space. This property will play a crucial role in the upcoming analysis.

3 Regularized nonlinear problem and its compatibility conditions

We construct the solution of problem (1.5) by taking the limit $\varepsilon \rightarrow +0$ in the following regularized problem.

$$(3.1) \quad \begin{cases} \mathbf{v}_t^\varepsilon = \mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon + \varepsilon \mathbf{v}_{ss}^\varepsilon + \varepsilon |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon, & s \in I, t > 0, \\ \mathbf{v}^\varepsilon(s, 0) = \mathbf{v}_0^\varepsilon(s), & s \in I, t > 0, \\ \mathbf{v}^\varepsilon(0, t) = \mathbf{a}, \mathbf{v}^\varepsilon(1, t) = \mathbf{e}_3, & t > 0. \end{cases}$$

From here on, it is assumed that $|\mathbf{v}_0| \equiv 1$ holds, i.e. the initial datum for the original problem (1.5) is parametrized by its arc length. The regularization shown above was chosen for two main reasons. Firstly, $\varepsilon \mathbf{v}_{ss}^\varepsilon$ was added so that the associated linear equation becomes a parabolic equation. Secondly, the term $\varepsilon |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon$ was added so that if the initial datum \mathbf{v}_0^ε satisfies $|\mathbf{v}_0^\varepsilon| \equiv 1$, then a solution of (3.1) also satisfies $|\mathbf{v}^\varepsilon| \equiv 1$.

Since we modified the equation, we must make corrections to the initial datum to insure that the compatibility conditions for problem (3.1) are satisfied.

3.1 Compatibility conditions for (3.1)

We first derive the compatibility conditions for problem (3.1). We set $\mathbf{Q}_0(\mathbf{v}) := \mathbf{v}$, $\mathbf{Q}_1(\mathbf{v}) := \mathbf{v} \times \mathbf{v}_{ss} + \varepsilon \mathbf{v}_{ss} + \varepsilon |\mathbf{v}_s|^2 \mathbf{v}$, and

$$\begin{aligned} \mathbf{Q}_m(\mathbf{v}) := & \sum_{j=0}^{m-1} \binom{m-1}{j} \mathbf{Q}_j \times \partial_s^2 \mathbf{Q}_{m-1-j}^\varepsilon + \varepsilon \partial_s^2 \mathbf{Q}_{m-1}(\mathbf{v}) \\ & + \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} \binom{m-1}{j} \binom{m-1-j}{k} (\partial_s \mathbf{Q}_j \cdot \partial_s \mathbf{Q}_k) \mathbf{Q}_{m-1-j-k} \end{aligned}$$

for $m \geq 2$. We arrive at the following compatibility conditions.

Definition 3.1 (*Compatibility conditions for (3.1)*). For $m \in \mathbf{N} \cup \{0\}$, we say that \mathbf{v}_0^ε satisfies the m -th order compatibility condition for (3.1) if $\mathbf{v}_0^\varepsilon \in H^{2m+1}(I)$ and

$$\mathbf{v}_0^\varepsilon(0) = \mathbf{a}, \quad \mathbf{v}_0^\varepsilon(1) = \mathbf{e}_3,$$

when $m = 0$, and

$$\mathbf{Q}_m(\mathbf{v}_0^\varepsilon(0)) = \mathbf{Q}_m(\mathbf{v}_0^\varepsilon(1)) = 0,$$

when $m \geq 1$. We also say that \mathbf{v}_0^ε satisfies the compatibility conditions for (3.1) up to order m if \mathbf{v}_0^ε satisfies the k -th order compatibility condition for all k with $0 \leq k \leq m$.

3.2 Corrections to the initial datum

Now we construct a corrected initial datum \mathbf{v}_0^ε such that given an initial datum \mathbf{v}_0 that satisfies the compatibility conditions for (1.5), \mathbf{v}_0^ε satisfies the compatibility conditions for (3.1) and $\mathbf{v}_0^\varepsilon \rightarrow \mathbf{v}_0$ in the appropriate function space. We must also correct the initial datum so that $|\mathbf{v}_0^\varepsilon| \equiv 1$ is satisfied.

Suppose that we have an initial datum $\mathbf{v}_0 \in H^{2m+1}(I)$ satisfying the compatibility conditions for (1.5) up to order m (the case when the initial datum belongs to a Sobolev space with even index will be remarked on at the end). We construct \mathbf{v}_0^ε in the form

$$(3.2) \quad \mathbf{v}_0^\varepsilon = \frac{\mathbf{v}_0 + \mathbf{h}^\varepsilon}{|\mathbf{v}_0 + \mathbf{h}^\varepsilon|},$$

where \mathbf{h}^ε is constructed so that $\mathbf{h}^\varepsilon \rightarrow \mathbf{0}$ as $\varepsilon \rightarrow +0$ in $H^{2m+1}(I)$. We do this by determining the differential coefficients of \mathbf{h}^ε at $s = 0, 1$ and extend it to I so that \mathbf{h}^ε belongs to $H^{2m+1}(I)$. We introduce some notations. We set

$$\mathbf{g}_0^\varepsilon(\mathbf{V}) := \mathbf{V},$$

$$\mathbf{g}_1^\varepsilon(\mathbf{V}) := \mathbf{V} \times \mathbf{V}_{ss} + \varepsilon \mathbf{V}_{ss} + \varepsilon |\mathbf{V}_s|^2 \mathbf{V},$$

$$\mathbf{g}_{m+1}^\varepsilon(\mathbf{V}) := D\mathbf{g}_m^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})],$$

where $m \geq 1$ and D is the derivative with respect to \mathbf{V} , i.e. $D\mathbf{g}_m^\varepsilon(\mathbf{V})[\mathbf{W}] = \frac{d}{dr}\mathbf{g}_m^\varepsilon(\mathbf{V} + r\mathbf{W})|_{r=0}$. Here, $|_{r=0}$ is the trace at $r = 0$. Under these notations, the m -th order compatibility condition for (3.1) can be expressed as $\mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon(0)) = \mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon(1)) = \mathbf{0}$ because $\mathbf{g}_m^\varepsilon(\mathbf{V}) = \mathbf{Q}_m(\mathbf{V})$. We gave a different notation because it is more convenient for the upcoming calculations. We first prove

Lemma 3.2 *If $|\mathbf{V}| \equiv 1$, then for any $m \geq 1$,*

$$(3.3) \quad \sum_{k=0}^m \binom{m}{k} \mathbf{g}_k^\varepsilon(\mathbf{V}) \cdot \mathbf{g}_{m-k}^\varepsilon(\mathbf{V}) \equiv 0.$$

Proof. We show this by induction. By direct calculation, we see that

$$\mathbf{V} \cdot \mathbf{g}_1^\varepsilon(\mathbf{V}) = \frac{\varepsilon}{2} (|\mathbf{V}|^2)_{ss} + \varepsilon |\mathbf{V}_s|^2 (|\mathbf{V}|^2 - 1) \equiv 0,$$

which proves (3.3) with $m = 1$. Suppose (3.3) holds up to some m with $m \geq 1$. From the assumption of induction, we have for any vector \mathbf{W} and $r \in \mathbf{R}$,

$$\sum_{k=0}^m \binom{m}{k} \mathbf{g}_k^\varepsilon \left(\frac{\mathbf{V} + r\mathbf{W}}{|\mathbf{V} + r\mathbf{W}|} \right) \cdot \mathbf{g}_{m-k}^\varepsilon \left(\frac{\mathbf{V} + r\mathbf{W}}{|\mathbf{V} + r\mathbf{W}|} \right) \equiv 0.$$

Differentiating with respect to r and setting $r = 0$ yields

$$\begin{aligned} \sum_{k=0}^m \binom{m}{k} \{ D\mathbf{g}_k^\varepsilon(\mathbf{V})[\mathbf{W} - (\mathbf{V} \cdot \mathbf{W})\mathbf{V}] \cdot \mathbf{g}_{m-k}^\varepsilon(\mathbf{V}) \\ + \mathbf{g}_k^\varepsilon(\mathbf{V}) \cdot D\mathbf{g}_{m-k}^\varepsilon(\mathbf{V})[\mathbf{W} - (\mathbf{V} \cdot \mathbf{W})\mathbf{V}] \} \equiv 0. \end{aligned}$$

By choosing $\mathbf{W} = \mathbf{g}_1^\varepsilon(\mathbf{V})$, we have

$$\begin{aligned} 0 &\equiv \sum_{k=0}^m \binom{m}{k} \{ \mathbf{g}_{k+1}^\varepsilon(\mathbf{V}) \cdot \mathbf{g}_{m-k}^\varepsilon(\mathbf{V}) + \mathbf{g}_k^\varepsilon(\mathbf{V}) \cdot \mathbf{g}_{m+1-k}^\varepsilon(\mathbf{V}) \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} \mathbf{g}_k^\varepsilon(\mathbf{V}) \cdot \mathbf{g}_{m+1-k}^\varepsilon(\mathbf{V}), \end{aligned}$$

which proves (3.3) for the case $m+1$, and this finishes the proof. \square .

Next, we make the following notations.

$$\mathbf{f}_0(\mathbf{V}) := \mathbf{V},$$

$$\mathbf{f}_1(\mathbf{V}) := \mathbf{V} \times \mathbf{V}_{ss},$$

$$\mathbf{f}_{m+1}(\mathbf{V}) := D\mathbf{f}_m(\mathbf{V})[\mathbf{f}_1(\mathbf{V})],$$

which is equivalent to taking $\varepsilon = 0$ in \mathbf{g}_m^ε . Hence,

$$\sum_{k=0}^m \binom{m}{k} \mathbf{f}_k(\mathbf{V}) \cdot \mathbf{f}_{m-k}(\mathbf{V}) \equiv 0$$

for any vector \mathbf{V} satisfying $|\mathbf{V}| \equiv 1$. Also, the m -th order compatibility condition for (1.5) can be expressed as $\mathbf{f}_m(\mathbf{v}_0(0)) = \mathbf{f}_m(\mathbf{v}_0(1)) = \mathbf{0}$ for $m \geq 1$.

From here we look into the structure of \mathbf{f}_m and \mathbf{g}_m^ε in more detail. This will allow us to determine the differential coefficients of the correction term \mathbf{h}^ε . We prove

Lemma 3.3 *For $m \geq 1$,*

$$(3.4) \quad \mathbf{g}_m^\varepsilon(\mathbf{V}) = \mathbf{f}_m(\mathbf{V}) + \varepsilon \mathbf{r}_m^\varepsilon(\mathbf{V}),$$

where $\mathbf{r}_1^\varepsilon(\mathbf{V}) := \mathbf{V}_{ss} + |\mathbf{V}_s|^2 \mathbf{V}$ and

$$\mathbf{r}_m^\varepsilon(\mathbf{V}) := D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] + D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]$$

for $m \geq 2$. Formula (3.4) tells us that the difference $\mathbf{g}_m^\varepsilon - \mathbf{f}_m$ is of order ε .

Proof. We prove (3.4) by induction. It is obvious that (3.4) holds for $m = 1$ from the definition of \mathbf{g}_1^ε , \mathbf{f}_1 , and \mathbf{r}_1^ε . Suppose it holds up to $m-1$ for some $m \geq 2$. From the assumption of induction, we have for any vector \mathbf{W} and $r \in \mathbf{R}$,

$$\mathbf{g}_{m-1}^\varepsilon(\mathbf{V} + r\mathbf{W}) = \mathbf{f}_{m-1}(\mathbf{V} + r\mathbf{W}) + \varepsilon \mathbf{r}_{m-1}^\varepsilon(\mathbf{V} + r\mathbf{W})$$

holds. Differentiating with respect to r and setting $r = 0$ yields

$$D\mathbf{g}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{W}] = D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{W}] + \varepsilon D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{W}].$$

Choosing $\mathbf{W} = \mathbf{g}_1^\varepsilon(\mathbf{V})$ yields

$$\begin{aligned}
\mathbf{g}_m^\varepsilon(\mathbf{V}) &= D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] + \varepsilon D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] \\
&= D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{f}_1(\mathbf{V}) + \varepsilon \mathbf{r}_1^\varepsilon(\mathbf{V})] + \varepsilon D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] \\
&= \mathbf{f}_m(\mathbf{V}) + \varepsilon \{ D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] + D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] \} \\
&= \mathbf{f}_m(\mathbf{V}) + \varepsilon \mathbf{r}_m^\varepsilon(\mathbf{V}),
\end{aligned}$$

which finishes the proof. □.

Next we prove

Lemma 3.4 For $m \geq 1$,

$$(3.5) \quad \mathbf{f}_m(\mathbf{V}) = A_m(\mathbf{V})\partial_s^{2m}\mathbf{V} + \phi_m(\mathbf{V})$$

holds, where the vector $\phi_m(\mathbf{V})$ and the operator $A_m(\mathbf{V})$ are defined as follows.

$$A_0(\mathbf{V})\mathbf{W} = \mathbf{W}, \quad A_1(\mathbf{V})\mathbf{W} = \mathbf{V} \times \mathbf{W}$$

$$\phi_1(\mathbf{V}) = \mathbf{0}.$$

and for $m \geq 2$,

$$A_m(\mathbf{V})\mathbf{W} = \mathbf{V} \times (A_{m-1}(\mathbf{V})\mathbf{W})$$

$$\begin{aligned}
\phi_m(\mathbf{V}) &= D\phi_{m-1}(\mathbf{V})[\mathbf{f}_1(\mathbf{V})] + A_{m-1}(\mathbf{V}) \sum_{j=0}^{2m-3} \binom{2(m-1)}{j} \partial_s^{2(m-1)-j}\mathbf{V} \times \partial_s^{j+2}\mathbf{V} \\
&\quad + (DA_m(\mathbf{V})[\mathbf{f}_1(\mathbf{V})]) \partial_s^{2(m-1)}\mathbf{V}.
\end{aligned}$$

Although we do not need A_0 for this lemma, we defined it because we will use it later. We also note that from the definition of ϕ_m , $\phi_m(\mathbf{V})$ satisfies

$$|\phi_m(\mathbf{V})| \leq C(|\mathbf{V}| + |\mathbf{V}_s| + \cdots + |\partial_s^{2m-1}\mathbf{V}|)$$

if $|\mathbf{V}| + |\mathbf{V}_s| + \cdots + |\partial_s^{2m-1}\mathbf{V}| \leq M$ and $C > 0$ depends on M . Hence, formula (3.5) gives the explicit form of the term with the highest order of s -derivatives in $\mathbf{f}_m(\mathbf{V})$.

Proof. We see that (3.5) holds for $m = 1$ from the definition of $\mathbf{f}_1(\mathbf{V})$. Suppose it holds up to $m - 1$ for some $m \geq 2$. Then for any vector \mathbf{W} and $r \in \mathbf{R}$ we have

$$\mathbf{f}_{m-1}(\mathbf{V} + r\mathbf{W}) = A_{m-1}(\mathbf{V} + r\mathbf{W})\partial_s^{2(m-1)}(\mathbf{V} + r\mathbf{W}) + \phi_{m-1}(\mathbf{V} + r\mathbf{W}).$$

Differentiating with respect to r , setting $r = 0$, and choosing $\mathbf{W} = \mathbf{f}_1(\mathbf{V})$ yields

$$\begin{aligned}
\mathbf{f}_m(\mathbf{V}) &= A_{m-1}(\mathbf{V})\partial_s^{2(m-1)}(\mathbf{f}_1(\mathbf{V})) \\
&\quad + (DA_{m-1}(\mathbf{V})[\mathbf{f}_1(\mathbf{V})])\partial_s^{2(m-1)}\mathbf{V} + D\phi_{m-1}(\mathbf{V})[\mathbf{f}_1(\mathbf{V})]
\end{aligned}$$

Since $\mathbf{f}_1(\mathbf{V}) = \mathbf{V} \times \mathbf{V}_{ss}$, we have

$$\begin{aligned}
A_{m-1}(\mathbf{V})\partial_s^{2(m-1)}(\mathbf{f}_1(\mathbf{V})) &= A_{m-1}(\mathbf{V})(\mathbf{V} \times \partial_s^{2m}\mathbf{V}) \\
&\quad + A_{m-1}(\mathbf{V}) \left\{ \sum_{j=0}^{2m-3} \binom{2(m-1)}{j} \partial_s^{2(m-1)-j}\mathbf{V} \times \partial_s^{j+2}\mathbf{V} \right\} \\
&= A_m(\mathbf{V})\partial_s^{2m}\mathbf{V} \\
&\quad + A_{m-1}(\mathbf{V}) \left\{ \sum_{j=0}^{2m-3} \binom{2(m-1)}{j} \partial_s^{2(m-1)-j}\mathbf{V} \times \partial_s^{j+2}\mathbf{V} \right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathbf{f}_m(\mathbf{V}) &= A_m(\mathbf{V})\partial_s^{2m}\mathbf{V} \\
&\quad + A_{m-1}(\mathbf{V}) \sum_{j=0}^{2m-3} \binom{2(m-1)}{j} \partial_s^{2(m-1)-j}\mathbf{V} \times \partial_s^{j+2}\mathbf{V} \\
&\quad + (DA_{m-1}(\mathbf{V})[\mathbf{f}_1(\mathbf{V})])\partial_s^{2(m-1)}\mathbf{V} + D\phi_{m-1}(\mathbf{V})[\mathbf{f}_1(\mathbf{V})] \\
&= A_m(\mathbf{V})\partial_s^{2m}\mathbf{V} + \phi_m(\mathbf{V}),
\end{aligned}$$

which finishes the proof. \square .

Next we prove

Lemma 3.5 For $m \geq 1$,

$$(3.6) \quad \mathbf{r}_m^\varepsilon(\mathbf{V}) = \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{V}) \partial_s^{2m}\mathbf{V} + \varphi_m^\varepsilon(\mathbf{V})$$

holds. Here, $\varphi_1^\varepsilon(\mathbf{V}) = |\mathbf{V}_s|^2\mathbf{V}$ and for $m \geq 2$,

$$\begin{aligned}
\varphi_m^\varepsilon(\mathbf{V}) &= D\varphi_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] + D\phi_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] \\
&\quad + \sum_{j=0}^{m-1} d_{m-1,j} \varepsilon^j A_{m-1-j}(\mathbf{V}) \{ \partial_s^{2(m-1)}(|\mathbf{V}_s|^2\mathbf{V}) \} \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) \left\{ \sum_{k=0}^{2m-3} \binom{2m-2}{k} \partial_s^{2m-2-k}\mathbf{V} \times \partial_s^{j+2}\mathbf{V} \right\} \\
&\quad + \sum_{j=0}^{m-1} d_{m-1,j} \varepsilon^j (DA_{m-1-j}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2m-2}\mathbf{V} \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V})[\mathbf{f}_1(\mathbf{V})]) \partial_s^{2m-2}\mathbf{V}.
\end{aligned}$$

$c_{m,j}$ and $d_{m,j}$ are constants defined as follows. $c_{1,0} = 1$ and

$$c_{m,j} = \begin{cases} m, & j = 0, \\ c_{m-1,j-1} + c_{m-1,j}, & 1 \leq j \leq m-2, \\ c_{m-1,m-2}, & j = m-1, \end{cases}$$

for $m \geq 2$. $d_{m,0} = 1$ and $d_{m,j} = c_{m,j-1}$ for $1 \leq j \leq m$. We also note that from the definition of φ_m^ε , $\varphi_m^\varepsilon(\mathbf{V})$ satisfies

$$|\varphi_m^\varepsilon(\mathbf{V})| \leq C(|\mathbf{V}| + |\mathbf{V}_s| + \cdots + |\partial_s^{2m-1}\mathbf{V}|)$$

if $|\mathbf{V}| + |\mathbf{V}_s| + \cdots + |\partial_s^{2m-1}\mathbf{V}| \leq M$ and $C > 0$ depends on M .

Proof. Since $\mathbf{r}_1^\varepsilon(\mathbf{V}) = \mathbf{V}_{ss} + |\mathbf{V}_s|^2\mathbf{V}$, (3.6) holds with $m = 1$. Suppose it holds up to $m-1$ for some $m \geq 2$. Then we have

$$\mathbf{r}_{m-1}^\varepsilon(\mathbf{V} + r\mathbf{W}) = \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V} + r\mathbf{W}) \partial_s^{2(m-1)}(\mathbf{V} + r\mathbf{W}) + \varphi_{m-1}^\varepsilon(\mathbf{V} + r\mathbf{W})$$

for any vector \mathbf{W} and $r \in \mathbf{R}$. Differentiating with respect to r , setting $r = 0$, and choosing $\mathbf{W} = \mathbf{g}_1^\varepsilon(\mathbf{V})$ yields

$$\begin{aligned} D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] &= \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) (\partial_s^{2(m-1)} \mathbf{g}_1^\varepsilon(\mathbf{V})) \\ &\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\ &\quad + D\varphi_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})]. \end{aligned}$$

Furthermore, we have proved that $\mathbf{f}_{m-1}(\mathbf{V}) = A_{m-1}(\mathbf{V}) \partial_s^{2(m-1)} \mathbf{V} + \phi_{m-1}(\mathbf{V})$, so that

$$\begin{aligned} D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] &= A_{m-1}(\mathbf{V}) (\partial_s^{2(m-1)} \mathbf{r}_1^\varepsilon(\mathbf{V})) + (DA_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\ &\quad + D\phi_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] \end{aligned}$$

holds. Substituting the definition of $\mathbf{r}_1^\varepsilon(\mathbf{V})$ into the first term on the right-hand side yields

$$\begin{aligned} D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] &= A_{m-1}(\mathbf{V}) \partial_s^{2m} \mathbf{V} + A_{m-1}(\mathbf{V}) (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})) \\ &\quad + (DA_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} + D\phi_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]. \end{aligned}$$

From the definition of $\mathbf{r}_m^\varepsilon(\mathbf{V})$, we have

$$\begin{aligned}
\mathbf{r}_m^\varepsilon(\mathbf{V}) &= D\mathbf{r}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] + D\mathbf{f}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] \\
&= \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) (\partial_s^{2(m-1)} \mathbf{g}_1^\varepsilon(\mathbf{V})) \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\
&\quad + D\boldsymbol{\varphi}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] + A_{m-1}(\mathbf{V}) \partial_s^{2m} \mathbf{V} + A_{m-1}(\mathbf{V}) (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})) \\
&\quad + (DA_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} + D\boldsymbol{\phi}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]
\end{aligned}$$

Since $\mathbf{g}_1^\varepsilon(\mathbf{V}) = \mathbf{V} \times \mathbf{V}_{ss} + \varepsilon(\mathbf{V}_{ss} + |\mathbf{V}_s|^2 \mathbf{V})$, we have

$$\begin{aligned}
\partial_s^{2(m-1)} \mathbf{g}_1^\varepsilon(\mathbf{V}) &= \mathbf{V} \times \partial_s^{2m} \mathbf{V} + \varepsilon \partial_s^{2m} \mathbf{V} + \sum_{j=0}^{2m-3} \binom{2m-2}{j} \partial_s^{2m-2-j} \mathbf{V} \times \partial_s^{j+2} \mathbf{V} \\
&\quad + \varepsilon (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})).
\end{aligned}$$

Substituting the above yields

$$\begin{aligned}
\mathbf{r}_m^\varepsilon(\mathbf{V}) &= \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) (\mathbf{V} \times \partial_s^{2m} \mathbf{V} + \varepsilon \partial_s^{2m} \mathbf{V}) + A_{m-1}(\mathbf{V}) \partial_s^{2m} \mathbf{V} \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) \left\{ \sum_{k=0}^{2m-3} \binom{2m-2}{k} \partial_s^{2(m-1)-k} \mathbf{V} \times \partial_s^{k+2} \mathbf{V} \right. \\
&\quad \left. + \varepsilon (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})) \right\} \\
&\quad + A_{m-1}(\mathbf{V}) (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})) + D\boldsymbol{\varphi}_{m-1}^\varepsilon(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})] + D\boldsymbol{\phi}_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})] \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V})[\mathbf{g}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\
&\quad + (DA_{m-1}(\mathbf{V})[\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V}.
\end{aligned}$$

The terms with the highest order of derivatives, i.e. terms containing $\partial_s^{2m} \mathbf{V}$, can be

further calculated as follows.

$$\begin{aligned}
& \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) (\mathbf{V} \times \partial_s^{2m} \mathbf{V} + \varepsilon \partial_s^{2m} \mathbf{V}) + A_{m-1}(\mathbf{V}) \partial_s^{2m} \mathbf{V} \\
&= \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-1-j}(\mathbf{V}) \partial_s^{2m} \mathbf{V} + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^{j+1} A_{m-2-j}(\mathbf{V}) \partial_s^{2m} \mathbf{V} + A_{m-1}(\mathbf{V}) \partial_s^{2m} \mathbf{V} \\
&= \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-1-j}(\mathbf{V}) \partial_s^{2m} \mathbf{V} + \sum_{j=1}^{m-1} c_{m-1,j-1} \varepsilon^j A_{m-1-j}(\mathbf{V}) \partial_s^{2m} \mathbf{V} + A_{m-1}(\mathbf{V}) \partial_s^{2m} \mathbf{V} \\
&= \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{V}) \partial_s^{2m} \mathbf{V},
\end{aligned}$$

where the definition of $c_{m,j}$ was used in the last equality. The other terms can also be calculated as follows.

$$\begin{aligned}
& \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j A_{m-2-j}(\mathbf{V}) \{ \varepsilon (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})) \} + A_{m-1}(\mathbf{V}) (\partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V})) \\
&= \sum_{j=0}^{m-1} d_{m-1,j} \varepsilon^j A_{m-1-j}(\mathbf{V}) \{ \partial_s^{2(m-1)} (|\mathbf{V}_s|^2 \mathbf{V}) \}, \\
& \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V}) [\mathbf{g}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} + (DA_{m-1}(\mathbf{V}) [\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\
&= \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V}) [\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} + (DA_{m-1}(\mathbf{V}) [\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V}) [\mathbf{f}_1(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\
&= \sum_{j=0}^{m-1} d_{m-1,j} \varepsilon^j (DA_{m-1-j}(\mathbf{V}) [\mathbf{r}_1^\varepsilon(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V} \\
&\quad + \sum_{j=0}^{m-2} c_{m-1,j} \varepsilon^j (DA_{m-2-j}(\mathbf{V}) [\mathbf{f}_1(\mathbf{V})]) \partial_s^{2(m-1)} \mathbf{V},
\end{aligned}$$

where $\mathbf{g}_1^\varepsilon(\mathbf{V}) = \mathbf{f}_1(\mathbf{V}) + \varepsilon \mathbf{r}_1^\varepsilon(\mathbf{V})$ was used. This finishes the proof. \square .

Now we are ready to construct \mathbf{h}^ε . We utilize the preceding lemmas to determine the

trace of \mathbf{h}^ε at $s = 0, 1$. First, we choose $\mathbf{h}^\varepsilon(0) = \mathbf{h}^\varepsilon(1) = \mathbf{0}$. This ensures that \mathbf{v}_0^ε satisfies the 0-th order compatibility condition.

From the explicit form of \mathbf{v}_0^ε we see that for $n \in \mathbf{N}$,

$$(3.7) \quad \partial_s^n \mathbf{v}_0^\varepsilon|_{s=0,1} = \partial_s^n \mathbf{v}_0 + \partial_s^n \mathbf{h}^\varepsilon - (\mathbf{v}_0 \cdot \partial_s^n \mathbf{h}^\varepsilon) \mathbf{v}_0 + \mathbf{q}_n(\mathbf{v}_0, \mathbf{h}^\varepsilon)|_{s=0,1}.$$

Here, $\mathbf{q}_n(\mathbf{v}_0, \mathbf{h}^\varepsilon)$ satisfies

$$|\mathbf{q}_n(\mathbf{v}_0, \mathbf{h}^\varepsilon)| \leq C(|\mathbf{h}^\varepsilon| + |\mathbf{h}_s^\varepsilon| + \cdots + |\partial_s^{n-1} \mathbf{h}^\varepsilon|)$$

if $|\mathbf{h}^\varepsilon| + |\mathbf{h}_s^\varepsilon| + \cdots + |\partial_s^{n-1} \mathbf{h}^\varepsilon| \leq M$, and $C > 0$ depends on \mathbf{v}_0 and M . From (3.7) and Lemma 3.4 we have

$$\mathbf{f}_m(\mathbf{v}_0^\varepsilon)|_{s=0,1} = \mathbf{f}_m(\mathbf{v}_0) + A_m(\mathbf{v}_0^\varepsilon)\{\partial_s^{2m} \mathbf{h}^\varepsilon - (\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0\} + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)|_{s=0,1},$$

for $m \geq 1$. Here, $\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)$ satisfies

$$|\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)| \leq C(|\mathbf{h}^\varepsilon| + |\mathbf{h}_s^\varepsilon| + \cdots + |\partial_s^{2m-1} \mathbf{h}^\varepsilon|)$$

if $|\mathbf{h}^\varepsilon| + |\mathbf{h}_s^\varepsilon| + \cdots + |\partial_s^{2m-1} \mathbf{h}^\varepsilon| \leq M$, and $C > 0$ depends on \mathbf{v}_0 and M . Since we chose $\mathbf{h}^\varepsilon|_{s=0,1} = \mathbf{0}$, it follows that $\mathbf{v}_0^\varepsilon|_{s=0,1} = \mathbf{v}_0|_{s=0,1}$, and thus we have

$$(3.8) \quad \mathbf{f}_m(\mathbf{v}_0^\varepsilon)|_{s=0,1} = \mathbf{f}_m(\mathbf{v}_0) + A_m(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)|_{s=0,1}.$$

Now we prove by induction that the differential coefficients of \mathbf{h}^ε at $s = 0, 1$ can be chosen so that \mathbf{v}_0^ε satisfies the compatibility conditions for (3.1) up to order m , and the differential coefficients of \mathbf{h}^ε at $s = 0, 1$ are $O(\varepsilon)$. We have already chosen $\mathbf{h}^\varepsilon|_{s=0,1} = \mathbf{0}$, which insures that \mathbf{v}_0^ε satisfies the 0-th order compatibility condition. Suppose that for a $m \geq 1$, the differential coefficients $\partial_s^j \mathbf{h}^\varepsilon|_{s=0,1}$ for $0 \leq j \leq 2(m-1)$ have been chosen in a way such that they are $O(\varepsilon)$ and \mathbf{v}_0^ε satisfies the compatibility conditions up to order $m-1$. First, we choose $\partial_s^{2m-1} \mathbf{h}^\varepsilon|_{s=0,1} = \mathbf{0}$. Then from Lemma 3.4, Lemma 3.5, and (3.8) we have

$$\begin{aligned} \mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1} &= \mathbf{f}_m(\mathbf{v}_0^\varepsilon) + \varepsilon \mathbf{r}_m^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1} \\ &= \mathbf{f}_m(\mathbf{v}_0) + A_m(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \\ &\quad + \varepsilon \left\{ \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{v}_0^\varepsilon + \varphi_m^\varepsilon(\mathbf{v}_0^\varepsilon) \right\} \Big|_{s=0,1} \\ &= A_m(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \\ &\quad + \varepsilon \left\{ \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{v}_0^\varepsilon + \varphi_m^\varepsilon(\mathbf{v}_0^\varepsilon) \right\} \Big|_{s=0,1}, \end{aligned}$$

where we have used the fact that \mathbf{v}_0 satisfies the m -th order compatibility condition for (1.5), i.e. $\mathbf{f}_m(\mathbf{v}_0)|_{s=0,1} = \mathbf{0}$. From (3.7), we have

$$\begin{aligned} \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{v}_0^\varepsilon &= \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{v}_0 + \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}_0^\varepsilon \\ &\quad - \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \{(\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0\} \\ &\quad + \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \mathbf{q}_{2m}(\mathbf{v}_0, \mathbf{h}_0^\varepsilon), \end{aligned}$$

which yields

(3.9)

$$\begin{aligned} \mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1} &= A_m(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \varepsilon \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \\ &\quad - c_{m,m-1} \varepsilon^m \{(\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0\} \\ &\quad + \varepsilon \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{v}_0 \\ &\quad + \varepsilon \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \mathbf{q}_{2m}(\mathbf{v}_0, \mathbf{h}_0^\varepsilon) \Big|_{s=0,1} \\ &=: A_m(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \varepsilon \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon \\ &\quad - c_{m,m-1} \varepsilon^m \{(\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0\} + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)|_{s=0,1}. \end{aligned}$$

We note that $\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)$ and $\mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)$ only contain terms with s -derivatives of \mathbf{v}_0 and \mathbf{h}^ε less than or equal to $2m - 1$. From Lemma 3.2, we see that

$$\sum_{k=0}^m \binom{m}{k} \mathbf{g}_k^\varepsilon(\mathbf{v}_0^\varepsilon) \cdot \mathbf{g}_{m-k}^\varepsilon(\mathbf{v}_0^\varepsilon) \equiv 0,$$

and the assumption of induction implies that $\mathbf{g}_k^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1} = \mathbf{0}$ for $1 \leq k \leq m - 1$. Hence we have

$$0 \equiv \mathbf{g}_0^\varepsilon(\mathbf{v}_0^\varepsilon) \cdot \mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1} = \mathbf{v}_0 \cdot \mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1}.$$

Substituting (3.9) into $\mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon)$ yields

$$\begin{aligned}
(3.10) \quad 0 &\equiv \mathbf{v}_0 \cdot \left\{ A_m(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon + \varepsilon \sum_{j=0}^{m-1} c_{m,j} \varepsilon^j A_{m-1-j}(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon \right. \\
&\quad \left. - c_{m,m-1} \varepsilon^m \{ (\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 \} + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right\} \Big|_{s=0,1} \\
&= \mathbf{v}_0 \cdot \left\{ \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right\} \Big|_{s=0,1}.
\end{aligned}$$

Furthermore, since

$$\mathbf{v}_0 \times (\mathbf{v}_0 \times \partial_s^{2m} \mathbf{h}^\varepsilon) = (\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 - \partial_s^{2m} \mathbf{h}^\varepsilon,$$

we see that

$$A_k(\mathbf{v}_0) \partial_s^{2m} \mathbf{h}^\varepsilon = \begin{cases} (-1)^l \mathbf{v}_0 \times \partial_s^{2m} \mathbf{h}^\varepsilon, & \text{when } k = 2l + 1, \\ (-1)^{l+1} ((\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 - \partial_s^{2m} \mathbf{h}^\varepsilon), & \text{when } k = 2l, \end{cases}$$

where $l \in \mathbf{N} \cup \{0\}$. From (3.9), we see that when $m = 2l + 1$,

$$\begin{aligned}
\mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon) \Big|_{s=0,1} &= (-1)^l \mathbf{v}_0 \times \partial_s^{2m} \mathbf{h}^\varepsilon + \varepsilon \left\{ \sum_{j=0}^l (-1)^{j+1} c_{m,2j} \varepsilon^{2j} ((\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 - \partial_s^{2m} \mathbf{h}^\varepsilon) \right. \\
&\quad \left. + \sum_{j=0}^{l-1} (-1)^j c_{m,2j+1} \varepsilon^{2j+1} \mathbf{v}_0 \times \partial_s^{2m} \mathbf{h}^\varepsilon \right\} \\
&\quad - c_{m,m-1} \varepsilon^m \{ (\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 \} + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \Big|_{s=0,1} \\
&= \left((-1)^l + \varepsilon \sum_{j=0}^{l-1} (-1)^j c_{m,2j+1} \varepsilon^{2j+1} \right) \mathbf{v}_0 \times \partial_s^{2m} \mathbf{h}^\varepsilon \\
&\quad + \left(\varepsilon \sum_{j=0}^l (-1)^j c_{m,2j} \varepsilon^{2j} \right) \partial_s^{2m} \mathbf{h}^\varepsilon \\
&\quad + \varepsilon \sum_{j=0}^l (-1)^{j+1} c_{m,2j} \varepsilon^{2j} (\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 - c_{m,m-1} \varepsilon^m \{ (\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon) \mathbf{v}_0 \} \\
&\quad + \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \Big|_{s=0,1}
\end{aligned}$$

holds. At this point, we choose

$$\partial_s^{2m} \mathbf{h}^\varepsilon|_{s=0,1} = \mathbf{v}_0 \times \{B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon))\}|_{s=0,1},$$

where $B(\mathbf{v}_0)$ is the 3×3 matrix given by

$$\left((-1)^l + \varepsilon \sum_{j=0}^{l-1} (-1)^j c_{m,2j+1} \varepsilon^{2j+1} \right) I_3 - \left(\varepsilon \sum_{j=0}^l (-1)^j c_{m,2j} \varepsilon^{2j} \right) A(\mathbf{v}_0),$$

where I_3 is the 3×3 identity matrix, and $A(\mathbf{v}_0)$ is the representation matrix of $A_1(\mathbf{v}_0)$. The inverse of $B(\mathbf{v}_0)$ exists for sufficiently small ε , where the smallness depends only on \mathbf{v}_0 . More precisely, there exists $\varepsilon_{*,1} > 0$ such that for any $\varepsilon \in (0, \varepsilon_{*,1}]$, the inverse matrix can be expressed as

$$B(\mathbf{v}_0)^{-1} = ((-1)^l + \varepsilon M_1) (I_3 + D(\mathbf{v}_0)),$$

where

$$M_1 = \sum_{j=0}^{l-1} (-1)^j c_{m,2j+1} \varepsilon^{2j+1},$$

$$M_2 = \left(\varepsilon \sum_{j=0}^l (-1)^j c_{m,2j} \varepsilon^{2j} \right) \left((-1)^l + \varepsilon M_1 \right)^{-1},$$

$$D(\mathbf{v}_0) = \sum_{k=1}^{\infty} M_2^k A(\mathbf{v}_0)^k.$$

Since $M_2 = O(\varepsilon)$ for small ε , the above infinite sum absolutely converges. Noting that by this choice of $\partial_s^{2m} \mathbf{h}^\varepsilon|_{s=0,1}$, we have $\mathbf{v}_0 \cdot \partial_s^{2m} \mathbf{h}^\varepsilon|_{s=0,1} = 0$, and we see that

$$\begin{aligned}
\left. \mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon) \right|_{s=0,1} &= ((-1)^l + M_1) \mathbf{v}_0 \times \left\{ \mathbf{v}_0 \times \left(B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \right) \right\} \\
&\quad + ((-1)^l + \varepsilon M_1) M_2 \mathbf{v}_0 \times \left\{ B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \right\} \\
&\quad + \left. \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right|_{s=0,1} \\
&= -(((-1)^l + M_1) (B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon))) \\
&\quad + ((-1)^l + \varepsilon M_1) M_2 A(\mathbf{v}_0) \{ B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \} \\
&\quad + ((-1)^l + M_1) \left\{ \mathbf{v}_0 \cdot \left(B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \right) \right\} \mathbf{v}_0 \\
&\quad + \left. \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right|_{s=0,1} \\
&= -B(\mathbf{v}_0) B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \\
&\quad + ((-1)^l + M_1) \left\{ \mathbf{v}_0 \cdot \left(B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \right) \right\} \mathbf{v}_0 \\
&\quad + \left. \mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right|_{s=0,1} \\
&= ((-1)^l + M_1) \left\{ \mathbf{v}_0 \cdot \left(B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \right) \right\} \mathbf{v}_0 \Big|_{s=0,1}.
\end{aligned}$$

Since $B(\mathbf{v}_0)^{-1} = I_3 + D(\mathbf{v}_0)$, we have

$$\begin{aligned}
&\left. \mathbf{v}_0 \cdot \left(B(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)) \right) \right|_{s=0,1} \\
&= \mathbf{v}_0 \cdot \left(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right) \\
&\quad + \mathbf{v}_0 \cdot D(\mathbf{v}_0) \left(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) \right) \Big|_{s=0,1}.
\end{aligned}$$

From (3.10), the first term on the right-hand side is 0. Furthermore, since

$$D(\mathbf{v}_0) = \sum_{k=1}^{\infty} M_2^k A(\mathbf{v}_0)^k = A(\mathbf{v}_0) \left(\sum_{k=0}^{\infty} M_2^{k+1} A(\mathbf{v}_0)^k \right),$$

and $A(\mathbf{v}_0)\mathbf{W} = \mathbf{v}_0 \times \mathbf{W}$ for any vector \mathbf{W} , the second term on the right-hand side is also 0. Hence we have

$$\mathbf{g}_m^\varepsilon(\mathbf{v}_0^\varepsilon)|_{s=0,1} = \mathbf{0},$$

which means that \mathbf{v}_0^ε satisfies the m -th order compatibility condition. From the assumption of induction, $\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon)$ is $O(\varepsilon)$ and $B(\mathbf{v}_0)^{-1}$ is $O(1)$, which implies that $\partial_s^{2m} \mathbf{h}^\varepsilon|_{s=0,1}$ is $O(\varepsilon)$. This finishes the proof for odd m . The case $m = 2l$ ($l \geq 1$) can be treated in the same way, and one can prove that there exists $\varepsilon_{*,2} > 0$ such that for any $\varepsilon \in (0, \varepsilon_{*,2}]$, choosing

$$\partial_s^{2m} \mathbf{h}^\varepsilon|_{s=0,1} = \tilde{B}(\mathbf{v}_0)^{-1}(\mathbf{F}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon) + \varepsilon \mathbf{G}_m(\mathbf{v}_0, \mathbf{h}^\varepsilon))|_{s=0,1}$$

is sufficient. Here,

$$\tilde{B}(\mathbf{v}_0) = \left((-1)^l + \varepsilon \sum_{j=0}^{l-1} (-1)^j c_{m,2j} \varepsilon^{2j} \right) I_3 - \left(\varepsilon \sum_{j=0}^{l-1} (-1)^j c_{m,2j+1} \varepsilon^{2j+1} \right) A(\mathbf{v}_0).$$

Finally, after setting $\varepsilon_* := \min\{\varepsilon_{*,1}, \varepsilon_{*,2}\}$, for any $\varepsilon \in (0, \varepsilon_*]$, we define \mathbf{h}^ε on I as

$$\mathbf{h}^\varepsilon(s) = \psi_0(s) \left(\sum_{j=0}^m \frac{\partial_s^{2j} \mathbf{h}^\varepsilon(0)}{(2j)!} s^{2j} \right) + \psi_1(s) \left(\sum_{j=0}^m \frac{\partial_s^{2j} \mathbf{h}^\varepsilon(1)}{(2j)!} (s-1)^{2j} \right),$$

where ψ_0 and ψ_1 are smooth cut-off functions satisfying

$$\psi_0(s) = 1, \quad \psi_1(s) = 0, \quad \text{for } s \in [0, \tfrac{1}{3}],$$

$$\psi_0(s) = 0, \quad \psi_1(s) = 1, \quad \text{for } s \in [\tfrac{2}{3}, 1].$$

Since the differential coefficients $\partial_s^{2j} \mathbf{h}^\varepsilon|_{s=0,1}$ ($0 \leq j \leq m$) were chosen to be $O(\varepsilon)$, $\mathbf{h}^\varepsilon \rightarrow \mathbf{0}$ as $\varepsilon \rightarrow +0$ in $H^{2m+1}(I)$ and as a consequence, $\mathbf{v}_0^\varepsilon \rightarrow \mathbf{v}_0$ in $H^{2m+1}(I)$. When $\mathbf{v}_0 \in H^{2m}(I)$, we can regard \mathbf{v}_0 as an initial datum belonging to $H^{2m-1}(I)$ and apply our arguments above and from the explicit form of \mathbf{h}^ε , we see that $\mathbf{v}_0^\varepsilon \rightarrow \mathbf{v}_0$ in $H^{2m}(I)$.

Furthermore, by a similar calculation and also utilizing the arguments given in Rauch and Massey [11], we can prove that for any $N \geq 1$, we can construct a smooth approximating series $\{\mathbf{v}_0^n\}_{n=1}^\infty$ such that $\mathbf{v}_0^n \in H^{2(m+N)+1}(I)$ ($n \in \mathbf{N}$), \mathbf{v}_0^n satisfies the compatibility conditions for (1.5) up to order $m+N$, and $\mathbf{v}_0^n \rightarrow \mathbf{v}_0$ in $H^{2m+1}(I)$ as $n \rightarrow \infty$. The same is true for initial datum belonging to a Sobolev space with even indices.

We summarize the conclusions of this subsection in the following propositions.

Proposition 3.6 *Let l be an arbitrary non-negative integer. For $\mathbf{v}_0 \in H^{l+3}(I)$ satisfying the compatibility conditions for (1.5) up to order $[\frac{l+3}{2}]$, there exists $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*]$, there exists a corrected initial datum \mathbf{v}_0^ε such that $|\mathbf{v}_0^\varepsilon| \equiv 1$, $\mathbf{v}_0^\varepsilon \in H^{l+3}(I)$, \mathbf{v}_0^ε satisfies the compatibility conditions for (3.1) up to order $[\frac{l+3}{2}]$, and*

$$\mathbf{v}_0^\varepsilon \rightarrow \mathbf{v}_0 \text{ in } H^{l+3}(I)$$

as $\varepsilon \rightarrow +0$. \mathbf{v}_0^ε also satisfies

$$\|\mathbf{v}_0^\varepsilon\|_{l+3} \leq C \|\mathbf{v}_0\|_{l+3},$$

where $C > 0$ is independent of ε , which follows from the convergence of $\{\mathbf{v}_0^\varepsilon\}_{0 < \varepsilon \leq \varepsilon_*}$.

Proposition 3.7 *Let l and N be non-negative integers. For $\mathbf{v}_0 \in H^{l+3}(I)$ satisfying the compatibility conditions for (1.5) up to order $\lfloor \frac{l+3}{2} \rfloor$, there exists $\{\mathbf{v}_0^n\}_{n=1}^\infty$ such that for any $n \in \mathbb{N}$, $|\mathbf{v}_0^n| \equiv 1$, $\mathbf{v}_0^n \in H^{l+3+N}(I)$, \mathbf{v}_0^n satisfies the compatibility conditions for (1.5) up to order $\lfloor \frac{l+3+N}{2} \rfloor$, and*

$$\mathbf{v}_0^n \rightarrow \mathbf{v}_0 \text{ in } H^{l+3}(I)$$

as $n \rightarrow \infty$.

Hence, by combining the above two propositions, we see that given a \mathbf{v}_0 satisfying the compatibility conditions for (1.5), we can construct a smoother initial datum satisfying the necessary compatibility conditions for (3.1).

4 Construction of the solution to (3.1)

We construct the solution of (3.1) based on an iteration scheme for the following linearized problem.

$$(4.1) \quad \begin{cases} \mathbf{u}_t = \varepsilon \mathbf{u}_{ss} + \mathbf{b} \times \mathbf{u}_{ss} + \mathbf{f}, & s \in I, t > 0, \\ \mathbf{u}(s, 0) = \mathbf{u}_0(s), & s \in I, \\ \mathbf{u}(0, t) = \mathbf{a}, \mathbf{u}(1, t) = \mathbf{e}_3, & t > 0, \end{cases}$$

where $\mathbf{u}_0 = \mathbf{u}_0(s)$, $\mathbf{b} = \mathbf{b}(s, t)$, and $\mathbf{f} = \mathbf{f}(s, t)$ are given vector valued functions. We make some notations to define the compatibility conditions for (4.1). Set $\mathbf{L}_1(\mathbf{u}, \mathbf{b}, \mathbf{f}) := \varepsilon \mathbf{u}_{ss} + \mathbf{b} \times \mathbf{u}_{ss} + \mathbf{f}$ and

$$\mathbf{L}_m := \varepsilon \partial_s^2 \mathbf{L}_{m-1} + \sum_{j=0}^{m-1} \binom{m-1}{j} \partial_t^j \mathbf{b} \times \partial_s^2 \mathbf{L}_{m-1-j} + \partial_t^{m-1} \mathbf{f},$$

for $m \geq 2$. Here, $\mathbf{L}_k = \mathbf{L}_k(\mathbf{u}, \mathbf{b}, \mathbf{f})$ for $1 \leq k \leq m$. The compatibility conditions are defined as follows.

Definition 4.1 *(Compatibility conditions for (4.1)). For a non-negative integer m , we say that $\mathbf{u}_0 \in H^{2m+1}(I)$, $\mathbf{b} \in Y_T^m(I)$, and $\mathbf{f} \in Z_T^m(I)$ satisfy the m -th order compatibility condition for (4.1) if*

$$\mathbf{u}_0(0) = \mathbf{a}, \mathbf{u}_0(1) = \mathbf{e}_3$$

when $m = 0$, and

$$\mathbf{L}_m(\mathbf{u}_0, \mathbf{b}, \mathbf{f})|_{s=0,1} = \mathbf{0}$$

when $m \geq 1$. We also say that \mathbf{u}_0, \mathbf{b} , and \mathbf{f} satisfy the compatibility conditions for (4.1) up to order m if they satisfy the k -th order compatibility condition for all k with $0 \leq k \leq m$.

Here, $Y_T^m(I)$ and $Z_T^m(I)$ are function spaces defined as follows.

$$Y_T^m(I) := \bigcap_{j=0}^m C^j([0, T]; H^{2(m-j)+1}(I))$$

$$Z_T^m(I) := \bigcap_{j=0}^m C^j([0, T]; H^{2(m-j)}(I)).$$

Fix $m \geq 1$. The solution of (3.1) is constructed by the following iteration scheme. We define \mathbf{u}^n as the solution of

$$(4.2) \quad \begin{cases} \mathbf{u}_t^n = \varepsilon \mathbf{u}_{ss} + \mathbf{u}^{n-1} \times \mathbf{u}_{ss}^n + \varepsilon |\mathbf{u}_s^{n-1}|^2 \mathbf{u}^{n-1}, & s \in I, \ t > 0, \\ \mathbf{u}^n(s, 0) = \mathbf{v}_0^\varepsilon(s), & s \in I, \\ \mathbf{u}^n(0, t) = \mathbf{a}, \ \mathbf{u}^n(1, t) = \mathbf{e}_3, & t > 0, \end{cases}$$

for $n \geq 2$. Here, \mathbf{v}_0^ε is the corrected initial datum obtained in Section 3. Now, we must define \mathbf{u}^1 appropriately so that the necessary compatibility conditions are satisfied at each step of the iteration. This is accomplished by choosing

$$\mathbf{u}^1(s, t) = \mathbf{v}_0^\varepsilon(s) + \sum_{j=1}^m \frac{t^j}{j!} \mathbf{Q}_j(\mathbf{v}_0^\varepsilon),$$

where \mathbf{Q}_j was defined in Section 3. From Proposition 3.6 and 3.7, we can assume that \mathbf{v}_0^ε is smooth, and thus we assume that \mathbf{v}_0^ε is smooth enough that $\mathbf{u}^1 \in Y_T^{m+1}(I)$. The solvability of (4.1), and as a consequence, the fact that $\{\mathbf{u}^n\}_{n=1}^\infty$ is well-defined, can be proved by utilizing the Sobolev–Slobodetskii space $W_2^{2m+1+\alpha, m+(1+\alpha)/2}(I \times [0, T])$ with $\alpha \in (0, \frac{1}{2})$ following the arguments in Section 2 of Nishiyama [8]. The convergence of $\{\mathbf{u}^n\}_{n=1}^\infty$ can also be proved in the same Sobolev–Slobodetskii space, based on the estimate of the solution for (4.1). Also see Solonnikov [12] for the definition and properties of Sobolev–Slobodetskii spaces.

The limit of $\{\mathbf{u}^n\}_{n=1}^\infty$ is the desired solution of (3.1) and we arrive at the following existence theorem for (3.1).

Proposition 4.2 *Let $m \geq 1$ be an integer and $T > 0$. There exists a unique solution $\mathbf{v}^\varepsilon \in Y_T^m(I)$ to (3.1) with a smooth initial datum \mathbf{v}_0^ε .*

5 Unique solvability of problem (1.5)

We construct the solution of (1.5) by taking the limit $\varepsilon \rightarrow +0$ in (3.1). We first derive estimates for the solution of (3.1) uniform in ε , and then prove the convergence of the solution as $\varepsilon \rightarrow +0$. In this section and for the rest of the paper, C denotes generic positive constants which may be different from line to line. What C depends on will be stated when ever it is necessary.

5.1 Uniform estimate of \mathbf{v}^ε with respect to ε

We first prove the following.

Lemma 5.1 *For any $m \geq 1$, a solution $\mathbf{v}^\varepsilon \in Y_T^m(I)$ of (3.1) satisfies $|\mathbf{v}^\varepsilon| = 1$ in $I \times [0, T]$.*

Proof. Setting $h(s, t) := |\mathbf{v}^\varepsilon(s, t)|^2 - 1$, we see that h satisfies

$$\begin{cases} h_t = \varepsilon h_{ss} + 2\varepsilon |\mathbf{v}_s^\varepsilon|^2 h, & s \in I, t > 0, \\ h(s, 0) = 0, & s \in I, \\ h(0, t) = h(1, t) = 0, & t > 0, \end{cases}$$

where $|\mathbf{a}| = |\mathbf{e}_3| = 1$ was used. From the Sobolev's embedding theorem,

$$\frac{1}{2} \frac{d}{dt} \|h\|^2 = \varepsilon(h, h_{ss}) + 2\varepsilon(h, |\mathbf{v}_s^\varepsilon|^2 h) \leq -\varepsilon \|h_s\|^2 + C \|h\|^2$$

holds. Here, $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_1$, and from Gronwall's inequality we see that $h = 0$ in $I \times [0, T]$, and this finishes the proof. \square

We introduce a property which will be used throughout this section. When $\mathbf{v}_s^\varepsilon \neq \mathbf{0}$, \mathbf{v}^ε , $\frac{\mathbf{v}_s^\varepsilon}{|\mathbf{v}_s^\varepsilon|}$, and $\frac{\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon}{|\mathbf{v}_s^\varepsilon|}$ form an orthonormal basis of \mathbf{R}^3 . From this we have for $n \geq 2$,

$$(5.3) \quad \mathbf{v}_s^\varepsilon \times \partial_s^n \mathbf{v}^\varepsilon = -[\mathbf{v}^\varepsilon \cdot \partial_s^n \mathbf{v}^\varepsilon] \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon + [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^n \mathbf{v}^\varepsilon] \mathbf{v}^\varepsilon.$$

The above equation is also true when $\mathbf{v}_s^\varepsilon = \mathbf{0}$. We also have from Lemma 5.1,

$$(5.4) \quad \mathbf{v}^\varepsilon \cdot \partial_s^n \mathbf{v}^\varepsilon = -\frac{1}{2} \sum_{j=1}^{n-1} \binom{n}{j} \partial_s^j \mathbf{v}^\varepsilon \cdot \partial_s^{n-j} \mathbf{v}^\varepsilon$$

for $n \geq 2$, and $\mathbf{v}^\varepsilon \cdot \mathbf{v}_s^\varepsilon = 0$. We further introduce two auxiliary lemmas which we will use to prove the uniform estimate.

Lemma 5.2 *For $m \geq 2$, we have*

$$(5.5) \quad \begin{aligned} \partial_t^m \mathbf{v}^\varepsilon &= \sum_{j=0}^m a_{m,j} \varepsilon^j A_{m-j} \partial_s^{2m} \mathbf{v}^\varepsilon + \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} e_{m,j,k} \varepsilon^j A_{m-1-j-k} \mathbf{v}_s^\varepsilon \times (A_k \partial_s^{2m-1} \mathbf{v}^\varepsilon) \\ &\quad + \sum_{j=1}^m b_{m,j} \varepsilon^{m+1-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m-1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_m^\varepsilon, \end{aligned}$$

where \mathbf{U}_m^ε are terms that can be estimated as

$$|\mathbf{U}_m^\varepsilon|_{s=0,1} \leq C \|\mathbf{v}_s^\varepsilon\|_{2(m-1)},$$

with $C > 0$ depending on $\|\mathbf{v}_s^\varepsilon\|_{2(m-2)}$, and $A_k = A_k(\mathbf{v}^\varepsilon)$ for $0 \leq k \leq m$ is defined in Lemma 3.4. $a_{m,j}$, $e_{m,j,k}$, and $b_{m,j}$ are absolute constants independent of ε , and $a_{m,0} = 1$.

Lemma 5.3 *For $m \geq 2$ we have*

$$(5.6) \quad \mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon|_{s=0,1} = (-1)^m \varepsilon A_{m-1} \{ \partial_t^{m-1} \mathbf{v}_{ss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \partial_t^{m-1} \mathbf{v}_s^\varepsilon) \mathbf{v}^\varepsilon \} \\ + \varepsilon \left\{ \sum_{l=0}^{m-2} (-1)^l A_{l+2} \partial_t^l \partial_s^{2(m-l)} \mathbf{v}^\varepsilon \right\} + \varepsilon \mathbf{W}_m^\varepsilon|_{s=0,1},$$

$$(5.7) \quad \partial_s^j \mathbf{v}^\varepsilon \cdot \partial_s^k \mathbf{v}^\varepsilon|_{s=0,1} = \varepsilon Y_{j,k}|_{s=0,1} \quad \text{for } j+k = 2(m-1)+1.$$

where \mathbf{W}_m^ε are terms satisfying the estimate

$$|\mathbf{W}_m^\varepsilon|_{s=0,1}| \leq C \|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$$

with $C > 0$ depending on $\|\mathbf{v}_s^\varepsilon\|_{2(m-2)}$, and $Y_{j,k}$ are terms satisfying the estimate

$$|Y_{j,k}|_{s=0,1}| \leq C \|\mathbf{v}_s^\varepsilon\|_{2(m-1)}^2,$$

when $j+k = 2(m-1)+1$. Note that from the equation in (3.1), the right-hand side of (5.6) can be estimated from above by $\varepsilon C \|\mathbf{v}_s^\varepsilon\|_{2m}$, where $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$.

Lemma 5.2 can be proved by induction in a similar way as Lemma 3.4 and Lemma 3.5 and hence the details are omitted.

Proof of Lemma 5.3 We prove (5.6) and (5.7) simultaneously by induction. From $\mathbf{v}_t^\varepsilon|_{s=0,1} = \mathbf{0}$, we have from the equation

$$\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon + \varepsilon \mathbf{v}_{ss}^\varepsilon + \varepsilon |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon|_{s=0,1} = \mathbf{0}.$$

Acting the exterior product $\mathbf{v}^\varepsilon \times$ from the left-side in the above equation along with (5.4) yields

$$(5.8) \quad \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon \mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon - |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon|_{s=0,1}.$$

Hence we have

$$\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon (\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon))|_{s=0,1}$$

and from (5.4) we have

$$\mathbf{v}^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -3\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = 3\varepsilon (\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon))|_{s=0,1},$$

which proves (5.7) for $m = 2$. From the equation in (3.1) and $\mathbf{v}_t^\varepsilon|_{s=0,1} = \mathbf{0}$, we have

$$\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon (\mathbf{v}_{ss}^\varepsilon + |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon)|_{s=0,1}.$$

Taking the t derivative of the above yields

$$\mathbf{v}^\varepsilon \times \mathbf{v}_{tss}^\varepsilon|_{s=0,1} = -\varepsilon \{ \mathbf{v}_{tss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon) \mathbf{v}^\varepsilon \}|_{s=0,1}.$$

On the other hand, by substituting the equation for \mathbf{v}^ε , we have

$$\begin{aligned}
\mathbf{v}^\varepsilon \times \mathbf{v}_{tss}^\varepsilon|_{s=0,1} &= \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon) + 2\mathbf{v}^\varepsilon \times (\mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) + \varepsilon \mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon \\
&\quad + 2\varepsilon(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon)\mathbf{v}^\varepsilon + \varepsilon \mathbf{W}_2^\varepsilon|_{s=0,1} \\
&= (\mathbf{v}^\varepsilon \cdot \partial_s^4 \mathbf{v}^\varepsilon)\mathbf{v}^\varepsilon - \partial_s^4 \mathbf{v}^\varepsilon + \varepsilon \mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon + 2\varepsilon(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon)\mathbf{v}^\varepsilon \\
&\quad + 2\varepsilon Y_{0,3} \mathbf{v}_s^\varepsilon + \varepsilon \mathbf{W}_2^\varepsilon|_{s=0,1}.
\end{aligned}$$

Since the exact form of \mathbf{W}_m^ε is not needed for the proof and the application of the lemma, \mathbf{W}_2^ε will denote the collection of terms satisfying the property stated in the lemma, and may change from line to line. For example, the term $2\varepsilon Y_{0,3} \mathbf{v}_s^\varepsilon$ can be included in $\varepsilon \mathbf{W}_2^\varepsilon$. Combining the above two equations yields

$$\begin{aligned}
&(\mathbf{v}^\varepsilon \cdot \partial_s^4 \mathbf{v}^\varepsilon)\mathbf{v}^\varepsilon - \partial_s^4 \mathbf{v}^\varepsilon + \varepsilon \mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon + 2\varepsilon(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon)\mathbf{v}^\varepsilon + \varepsilon \mathbf{W}_2^\varepsilon|_{s=0,1} \\
&= -\varepsilon\{\mathbf{v}_{tss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon)\mathbf{v}^\varepsilon\}|_{s=0,1}.
\end{aligned}$$

Taking the exterior product with \mathbf{v}^ε from the left-side and rearranging the terms yield

$$\begin{aligned}
\mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon|_{s=0,1} &= \varepsilon \mathbf{v}^\varepsilon \times \{\mathbf{v}_{tss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon)\mathbf{v}^\varepsilon\} + \varepsilon \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon) + \varepsilon \mathbf{W}_2^\varepsilon|_{s=0,1} \\
&= \varepsilon A_1\{\mathbf{v}_{tss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon)\mathbf{v}^\varepsilon\} + \varepsilon \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon) + \varepsilon \mathbf{W}_2^\varepsilon|_{s=0,1},
\end{aligned}$$

which proves (5.6) with $m = 2$. Suppose that (5.6) and (5.7) hold up to $m - 1$ for some $m \geq 3$. From the assumption of induction, we have

$$\begin{aligned}
\mathbf{v}^\varepsilon \times \partial_s^{2(m-1)} \mathbf{v}^\varepsilon|_{s=0,1} &= (-1)^{m-1} \varepsilon A_{m-2} \{\partial_t^{m-2} \mathbf{v}_{ss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \partial_t^{m-2} \mathbf{v}_s^\varepsilon) \mathbf{v}^\varepsilon\} \\
&\quad + \varepsilon \left\{ \sum_{l=0}^{m-3} (-1)^l A_{l+2} \partial_t^l \partial_s^{2(m-1-l)} \mathbf{v}^\varepsilon \right\} + \varepsilon \mathbf{W}_{m-1}^\varepsilon|_{s=0,1}.
\end{aligned}$$

Taking the t derivative of the above yields

$$\begin{aligned}
(5.9) \quad \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)} \mathbf{v}_t^\varepsilon|_{s=0,1} &= (-1)^{m-1} \varepsilon A_{m-2} \{\partial_t^{m-1} \mathbf{v}_{ss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \partial_t^{m-1} \mathbf{v}_s^\varepsilon) \mathbf{v}^\varepsilon\} \\
&\quad + \varepsilon \left\{ \sum_{l=0}^{m-3} (-1)^l A_{l+2} \partial_t^{l+1} \partial_s^{2(m-1-l)} \mathbf{v}^\varepsilon \right\} + \varepsilon \mathbf{W}_m^\varepsilon|_{s=0,1},
\end{aligned}$$

where \mathbf{W}_m^ε denotes the collection of terms satisfying the condition of the lemma and as before, the contents of \mathbf{W}_m^ε will change from line to line. First we observe

$$\begin{aligned}
\partial_s^{2(m-1)} \mathbf{v}_t^\varepsilon &= \mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon + 2(m-1) \mathbf{v}_s^\varepsilon \times \partial_s^{2m-1} \mathbf{v}^\varepsilon + \varepsilon \partial_s^{2m} \mathbf{v}^\varepsilon + 2\varepsilon(\mathbf{v}_s^\varepsilon \cdot \partial_s^{2m-1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \\
&\quad + \sum_{j=2}^{2(m-1)-2} \binom{2(m-1)}{j} \partial_s^j \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon + \varepsilon \mathbf{W}_m^\varepsilon.
\end{aligned}$$

Taking the exterior product with \mathbf{v}^ε from the left yields

(5.10)

$$\begin{aligned} \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)} \mathbf{v}_t^\varepsilon &= (\mathbf{v}^\varepsilon \cdot \partial_s^{2m} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - \partial_s^{2m} \mathbf{v}^\varepsilon + 2(m-1)(\mathbf{v}^\varepsilon \cdot \partial_s^{2m-1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon + \varepsilon \mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon \\ &\quad + \sum_{j=2}^{2(m-1)-2} \binom{2(m-1)}{j} \mathbf{v}^\varepsilon \times (\partial_s^j \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon) + \varepsilon \mathbf{W}_m^\varepsilon. \end{aligned}$$

Since the terms in the summation can be calculated as

$$\mathbf{v}^\varepsilon \times (\partial_s^j \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon) = (\mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - (\mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon,$$

if j is odd, then $2(m-1)-j+2$ is also odd and we have from the assumption of induction for (5.7) that

$$\mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon|_{s=0,1} = \varepsilon Y_{0,2(m-1)-j+2}|_{s=0,1}$$

$$\mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon|_{s=0,1} = \varepsilon Y_{0,j}|_{s=0,1}$$

holds, which implies that the terms in the summation of (5.10) with odd j can be included into $\varepsilon \mathbf{W}_m^\varepsilon$. When j is even, $2(m-1)-j+2$ is also even and we have

$$\begin{aligned} &\partial_s^j \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon \\ &= \{(\mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^j \mathbf{v}^\varepsilon)\} \\ &\quad \times \{(\mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon)\} \\ &= -(\mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \times \{\mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon)\} \\ &\quad - (\mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon) [\mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^j \mathbf{v}^\varepsilon)] \times \mathbf{v}^\varepsilon \\ &\quad + [\mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^j \mathbf{v}^\varepsilon)] \times [\mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^{2(m-1)-j+2} \mathbf{v}^\varepsilon)], \end{aligned}$$

where we have used the decomposition

$$\mathbf{W} = (\mathbf{v}^\varepsilon \cdot \mathbf{W}) \mathbf{v}^\varepsilon - \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \mathbf{W})$$

for any vector \mathbf{W} , which follows from $|\mathbf{v}^\varepsilon| \equiv 1$. From the assumption of induction for (5.6), we see that $\mathbf{v}^\varepsilon \times \partial_s^k \mathbf{v}^\varepsilon$ can be included into $\varepsilon \mathbf{W}_m^\varepsilon$ for $k = j$ and $k = 2(m-1)-j+2$, and thus the terms in the summation of (5.10) with even j can also be included into $\varepsilon \mathbf{W}_m^\varepsilon$. Hence, we have

$$\begin{aligned} \mathbf{v}^\varepsilon \times \partial_s^{2(m-1)} \mathbf{v}_t^\varepsilon &= (\mathbf{v}^\varepsilon \cdot \partial_s^{2m} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - \partial_s^{2m} \mathbf{v}^\varepsilon + 2(m-1)(\mathbf{v}^\varepsilon \cdot \partial_s^{2m-1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon + \varepsilon \mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon \\ &\quad + \varepsilon \mathbf{W}_m^\varepsilon. \end{aligned}$$

Substituting the above to (5.9), taking the exterior product with \mathbf{v}^ε from the left, and multiplying by -1 yields

$$\begin{aligned}
\mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon|_{s=0,1} &= (-1)^m \varepsilon A_{m-1} \{ \partial_t^{m-1} \mathbf{v}_{ss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \partial_t^{m-1} \mathbf{v}_s^\varepsilon) \mathbf{v}^\varepsilon \} \\
&\quad + \varepsilon \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon) + \varepsilon \left\{ \sum_{l=0}^{m-3} (-1)^{l+1} A_{l+3} \partial_t^{l+1} \partial_s^{2(m-1-l)} \mathbf{v}^\varepsilon \right\} \\
&\quad + \varepsilon \mathbf{W}_m^\varepsilon|_{s=0,1} \\
&= (-1)^m \varepsilon A_{m-1} \{ \partial_t^{m-1} \mathbf{v}_{ss}^\varepsilon + 2(\mathbf{v}_s^\varepsilon \cdot \partial_t^{m-1} \mathbf{v}_s^\varepsilon) \mathbf{v}^\varepsilon \} \\
&\quad + \varepsilon \left\{ \sum_{l=0}^{m-2} A_{l+2} \partial_t^l \partial_s^{2(m-l)} \mathbf{v}^\varepsilon \right\} + \varepsilon \mathbf{W}_m^\varepsilon|_{s=0,1},
\end{aligned}$$

and this proves (5.6). Set the right-hand side of (5.6) as $\varepsilon \mathbf{Z}_m^\varepsilon|_{s=0,1}$.

To prove (5.7), take j and k such that $j + k = 2(m-1) + 1$. We assume without loss of generality that j is even and set $j = 2l$, which gives $k = 2(m-1-l) + 1$. We first consider the case $j \neq 0$. When $l = 1$, we have

$$\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon(\mathbf{v}_{ss}^\varepsilon + |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon)$$

Taking the exterior product with $\partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon$ from the left yields

$$\partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon)|_{s=0,1} = \varepsilon \partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon \times (\mathbf{v}_{ss}^\varepsilon + |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon)|_{s=0,1}.$$

Expanding the exterior products and rearranging the terms yield

$$\begin{aligned}
(\mathbf{v}_{ss}^\varepsilon \cdot \partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon|_{s=0,1} &= (\mathbf{v}^\varepsilon \cdot \partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon) \mathbf{v}_{ss}^\varepsilon + \varepsilon \partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon \times (\mathbf{v}_{ss}^\varepsilon + |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon)|_{s=0,1} \\
&= \varepsilon Y_{0,2(m-2)+1} \mathbf{v}_{ss}^\varepsilon + \varepsilon \partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon \times (\mathbf{v}_{ss}^\varepsilon + |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon)|_{s=0,1}.
\end{aligned}$$

Taking the inner product with \mathbf{v}^ε tells us that

$$\begin{aligned}
&\mathbf{v}_{ss}^\varepsilon \cdot \partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon|_{s=0,1} \\
&= \varepsilon \{ Y_{0,2(m-2)+1} (\mathbf{v}^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) + \mathbf{v}^\varepsilon \cdot (\partial_s^{2(m-2)+1} \mathbf{v}^\varepsilon \times (\mathbf{v}_{ss}^\varepsilon + |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon)) \}|_{s=0,1},
\end{aligned}$$

and setting $Y_{2,2(m-2)+1}$ as the expression in brackets in the right-hand side proves the case $l = 1$. From (5.6) we have for $2 \leq l \leq m-1$,

$$\mathbf{v}^\varepsilon \times \partial_s^{2l} \mathbf{v}^\varepsilon|_{s=0,1} = \varepsilon \mathbf{Z}_l^\varepsilon|_{s=0,1},$$

and taking the exterior product with $\partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon$ from the left yields

$$\partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^{2l} \mathbf{v}^\varepsilon)|_{s=0,1} = \varepsilon \partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon \times \mathbf{Z}_l^\varepsilon|_{s=0,1}.$$

Expanding the exterior product as before, we have

$$(\partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon \cdot \partial_s^{2l} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon|_{s=0,1} = \varepsilon \{ Y_{0,2(m-1-l)+1} \partial_s^{2l} \mathbf{v}^\varepsilon + \partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon \times \mathbf{Z}_l^\varepsilon \}|_{s=0,1}.$$

After taking the inner product of the above with \mathbf{v}^ε , we see that setting

$$Y_{2l,2(m-1-l)+1} := Y_{0,2(m-1-l)+1}(\mathbf{v}^\varepsilon \cdot \partial_s^{2l} \mathbf{v}^\varepsilon) + \mathbf{v}^\varepsilon \cdot (\partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon \times \mathbf{Z}_l^\varepsilon)$$

proves the case $l \geq 2$. Finally, when $l = 0$ we have from (5.4) that

$$\begin{aligned} \mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1)+1} \mathbf{v}^\varepsilon|_{s=0,1} &= -\frac{1}{2} \sum_{i=1}^{2(m-1)} \binom{2(m-1)+1}{i} \partial_s^i \mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1)+1-i} \mathbf{v}^\varepsilon|_{s=0,1} \\ &= -\frac{1}{2} \sum_{l=1}^{m-1} \binom{2(m-1)+1}{2l} \partial_s^{2l} \mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1-l)+1} \mathbf{v}^\varepsilon \\ &\quad - \frac{1}{2} \sum_{l=1}^{m-1} \binom{2(m-1)+1}{2l-1} \partial_s^{2l-1} \mathbf{v}^\varepsilon \cdot \partial_s^{2(m-1-l)} \mathbf{v}^\varepsilon|_{s=0,1} \\ &= \varepsilon \left\{ -\frac{1}{2} \sum_{l=1}^{m-1} \binom{2(m-1)+1}{2l} Y_{2l,2(m-1-l)+1} \right. \\ &\quad \left. - \frac{1}{2} \sum_{l=1}^{m-1} \binom{2(m-1)+1}{2l-1} Y_{2l-1,2(m-1-l)} \right\}|_{s=0,1}, \end{aligned}$$

which shows that

$$\begin{aligned} Y_{0,2(m-1)+1} &:= -\frac{1}{2} \sum_{l=1}^{m-1} \binom{2(m-1)+1}{2l} Y_{2l,2(m-1-l)+1} \\ &\quad - \frac{1}{2} \sum_{l=1}^{m-1} \binom{2(m-1)+1}{2l-1} Y_{2l-1,2(m-1-l)} \end{aligned}$$

proves the case $l = 0$ and this finishes the proof of (5.7) and the proof of the lemma. \square .

Now we prove the following uniform estimate.

Proposition 5.4 *Let $m \geq 1$ be an integer, $T > 0$, and $\mathbf{v}^\varepsilon \in Y_T^{m+2}(I)$ be the solution of (3.1). There exist $T_0 \in (0, T]$ and $\varepsilon_0 \in (0, \varepsilon_*]$ such that for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\|\mathbf{v}^\varepsilon\|_{Y_{T_0}^m(I)} \leq c_*,$$

where $c_* > 0$ depends on $\|\mathbf{v}_{0s}\|_{2m}$ and T_0 , and T_0 depends on $\|\mathbf{v}_{0s}\|_2$. Here, $\varepsilon_* > 0$ is given in Proposition 3.6.

Note that the assumption $\mathbf{v}^\varepsilon \in Y_T^{m+2}(I)$ is not essential because we can approximate the initial datum as smooth as we need and obtain solutions as smooth as necessary to justify the calculations below.

Proof. By the interpolation inequality and integration by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_s^\varepsilon\|^2 &= (\mathbf{v}_s^\varepsilon, \mathbf{v}_{st}^\varepsilon) = -(\mathbf{v}_{ss}^\varepsilon, \mathbf{v}_t^\varepsilon) + [\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_t^\varepsilon]_{s=0}^1 \\ &= -\varepsilon \|\mathbf{v}_{ss}^\varepsilon\|^2 - \varepsilon (\mathbf{v}_{ss}^\varepsilon, |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon) \\ &\leq -\varepsilon \|\mathbf{v}_{ss}^\varepsilon\|^2 + \frac{\varepsilon}{2} \|\mathbf{v}_s^\varepsilon\|_1^2 + C \|\mathbf{v}_s^\varepsilon\|^6, \end{aligned}$$

where $[\cdot]_{s=0}^1$ is defined by $[f]_{s=0}^1 = f|_{s=1} - f|_{s=0}$, and we have used $\mathbf{v}_t^\varepsilon|_{s=0,1} = \mathbf{0}$. We further calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{ss}^\varepsilon\|^2 &= -(\mathbf{v}_{sss}^\varepsilon, \mathbf{v}_{ts}^\varepsilon) + [\mathbf{v}_{ss}^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon]_{s=0}^1 \\ &= -(\mathbf{v}_{sss}^\varepsilon, \mathbf{v}_s^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) - \varepsilon \|\mathbf{v}_{sss}^\varepsilon\|^2 - 2\varepsilon (\mathbf{v}_{sss}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}^\varepsilon) \\ &\quad + -\varepsilon (\mathbf{v}_{sss}^\varepsilon, |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon) + [\mathbf{v}_{ss}^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon]_{s=0}^1. \end{aligned}$$

From equation (5.3) and (5.4), we have

$$\begin{aligned} \mathbf{v}_{sss}^\varepsilon \cdot (\mathbf{v}_s^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) &= -\mathbf{v}_{ss}^\varepsilon \cdot (\mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) \\ &= -\mathbf{v}_{ss}^\varepsilon \cdot \left\{ -(\mathbf{v}^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon + [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \mathbf{v}_{sss}^\varepsilon] \mathbf{v}^\varepsilon \right\} \\ &= -\mathbf{v}_{ss}^\varepsilon \cdot \left\{ 3(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon + [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon] \right\} \\ &= -3(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}_{ss}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) + |\mathbf{v}_s^\varepsilon|^2 (\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \mathbf{v}_{sss}^\varepsilon, \end{aligned}$$

which allows us to further calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{ss}^\varepsilon\|^2 &\leq C \|\mathbf{v}_s^\varepsilon\|_1^2 (1 + \|\mathbf{v}_s^\varepsilon\|_1^2) - \frac{\varepsilon}{2} \|\mathbf{v}_{sss}^\varepsilon\|^2 - (\mathbf{v}_{sss}^\varepsilon, |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) + [\mathbf{v}_{ss}^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon]_{s=0}^1 \\ &= C \|\mathbf{v}_s^\varepsilon\|^2 (1 + \|\mathbf{v}_s^\varepsilon\|^2) - \frac{\varepsilon}{2} \|\mathbf{v}_{sss}^\varepsilon\|^2 + 2(\mathbf{v}_{ss}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \\ &\quad + [|\mathbf{v}_s^\varepsilon|^2 \mathbf{v}_{ss}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon)]_{s=0}^1 + [\mathbf{v}_{ss}^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon]_{s=0}^1. \end{aligned}$$

We have seen that

$$\mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon \mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon - |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon|_{s=0,1}.$$

Hence, we have

$$\begin{aligned} |[\mathbf{v}_s^\varepsilon|^2 \mathbf{v}_{ss}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon)]_{s=0}^1| &= |[-\varepsilon |\mathbf{v}_s^\varepsilon|^2 (\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon)]_{s=0}^1| \\ &\leq \frac{\varepsilon}{8} \|\mathbf{v}_{ss}^\varepsilon\|_1^2 + C \|\mathbf{v}_s^\varepsilon\|_1^6 \end{aligned}$$

Furthermore, we have

$$\mathbf{v}_{ts}^\varepsilon = \mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon + \varepsilon \mathbf{v}_{sss}^\varepsilon + 2\varepsilon(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}^\varepsilon + \varepsilon |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}_s^\varepsilon,$$

and combining with (5.3), (5.4), and (5.8), we have

$$\mathbf{v}_{ss}^\varepsilon \cdot \mathbf{v}_{ts}^\varepsilon|_{s=0,1} = -\varepsilon(\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) - \varepsilon^2(\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) \cdot \mathbf{v}_{sss}^\varepsilon + \boldsymbol{\eta}_2^\varepsilon|_{s=0,1}$$

where $\boldsymbol{\eta}_2^\varepsilon$ are terms which can be estimated as

$$|\boldsymbol{\eta}_2^\varepsilon|_{s=0,1}| \leq \frac{\varepsilon}{8} \|\mathbf{v}_s^\varepsilon\|_2^2 + \|\mathbf{v}_s^\varepsilon\|_1^6,$$

and we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{ss}^\varepsilon\|^2 &\leq C \|\mathbf{v}_s^\varepsilon\|_1^2 (1 + \|\mathbf{v}_s^\varepsilon\|_1^4) - \frac{\varepsilon}{4} \|\mathbf{v}_{sss}^\varepsilon\|^2 \\ &\quad - [\varepsilon(\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)]_{s=0}^1 - [\varepsilon^2(\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) \cdot \mathbf{v}_{sss}^\varepsilon]_{s=0}^1. \end{aligned}$$

We further estimate $\mathbf{v}_{sss}^\varepsilon$ to close the estimate.

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{sss}^\varepsilon\|^2 &= -(\partial_s^4 \mathbf{v}^\varepsilon, \mathbf{v}_{tss}^\varepsilon) + [\mathbf{v}_{sss}^\varepsilon \cdot \mathbf{v}_{tss}^\varepsilon]_{s=0}^1 \\ &= -2(\partial_s^4 \mathbf{v}^\varepsilon, \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) - \varepsilon \|\partial_s^4 \mathbf{v}^\varepsilon\|^2 - \varepsilon(\partial_s^4 \mathbf{v}^\varepsilon, |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}_{ss}^\varepsilon) \\ &\quad - 2\varepsilon(\partial_s^4 \mathbf{v}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon) - 2\varepsilon(\partial_s^4 \mathbf{v}^\varepsilon, |\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon) - 4\varepsilon(\partial_s^4 \mathbf{v}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}_s^\varepsilon) \\ &\quad + [\mathbf{v}_{sss}^\varepsilon \cdot \mathbf{v}_{tss}^\varepsilon]_{s=0}^1. \end{aligned}$$

From (5.3) and (5.4), we see that

$$\begin{aligned} (\partial_s^4 \mathbf{v}^\varepsilon, \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) &= -4(\mathbf{v}_{sss}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) - 3(\mathbf{v}_{sss}^\varepsilon, |\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \\ &\quad + 3(\mathbf{v}_{ss}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^4 \mathbf{v}^\varepsilon] \mathbf{v}_s^\varepsilon), \end{aligned}$$

which yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{sss}^\varepsilon\|^2 &\leq C \|\mathbf{v}_s^\varepsilon\|_2^2 (1 + \|\mathbf{v}_s^\varepsilon\|_2^2) - \frac{\varepsilon}{2} \|\partial_s^4 \mathbf{v}^\varepsilon\|^2 - 6(\mathbf{v}_{ss}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^4 \mathbf{v}^\varepsilon] \mathbf{v}_s^\varepsilon) \\ &\quad + [\mathbf{v}_{sss}^\varepsilon \cdot \mathbf{v}_{tss}^\varepsilon]_{s=0}^1 \\ &\leq C \|\mathbf{v}_s^\varepsilon\|_2^2 (1 + \|\mathbf{v}_s^\varepsilon\|_2^2) - \frac{\varepsilon}{2} \|\partial_s^4 \mathbf{v}^\varepsilon\|^2 - 6[(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \mathbf{v}_{sss}^\varepsilon]]_{s=0}^1 \\ &\quad + [\mathbf{v}_{sss}^\varepsilon \cdot \mathbf{v}_{tss}^\varepsilon]_{s=0}^1, \end{aligned}$$

where integration by parts was used. We see from (5.8) and $\mathbf{v}^\varepsilon \cdot \mathbf{v}_s^\varepsilon \equiv 0$ that

$$\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon \mathbf{v}_s^\varepsilon \cdot (\mathbf{v}_s^\varepsilon \times \mathbf{v}_{ss}^\varepsilon)|_{s=0,1}$$

holds, and hence

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{sss}^\varepsilon\|^2 \leq C \|\mathbf{v}_s^\varepsilon\|_2^2 (1 + \|\mathbf{v}_s^\varepsilon\|_2^6) - \frac{\varepsilon}{4} \|\partial_s^4 \mathbf{v}^\varepsilon\|^2 + [\mathbf{v}_{sss}^\varepsilon \cdot \mathbf{v}_{tss}^\varepsilon]_{s=0}^1.$$

We further investigate the boundary term. From $\mathbf{v}_{tt}^\varepsilon|_{s=0,1} = \mathbf{0}$, we have

$$\begin{aligned} & \mathbf{v}^\varepsilon \times (\mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon) + 2\varepsilon \mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon + \varepsilon^2 \partial_s^4 \mathbf{v}^\varepsilon + 2\varepsilon \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon \\ & + 2\varepsilon^2 (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + 2\varepsilon [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon + \varepsilon \mathbf{w}_2^\varepsilon|_{s=0,1} = \mathbf{0}, \end{aligned}$$

from which we further deduce that

$$\begin{aligned} (1 - \varepsilon^2) \partial_s^4 \mathbf{v}^\varepsilon - 2\varepsilon \mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon|_{s=0,1} &= -4(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon - 3|\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon - 6(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}_s^\varepsilon \\ &+ 2\varepsilon \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon + 4\varepsilon^2 (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + 2\varepsilon [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon + \varepsilon \mathbf{w}_2^\varepsilon|_{s=0,1} \end{aligned}$$

where \mathbf{w}_2^ε are terms that can be estimated as

$$|\mathbf{w}_2^\varepsilon|_{s=0,1}| \leq C \|\mathbf{v}_s^\varepsilon\|_2 (1 + \|\mathbf{v}_s^\varepsilon\|_2^2),$$

and the exact form may change from line to line. Since we have $|\mathbf{v}^\varepsilon| \equiv 1$, there exists $\varepsilon_1 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_1]$, the matrix $I_3 - \frac{2\varepsilon}{1-\varepsilon^2} A(\mathbf{v}^\varepsilon)$ is reversible and the inverse matrix can be expressed as

$$\left(I_3 - \frac{2\varepsilon}{1-\varepsilon^2} A(\mathbf{v}^\varepsilon) \right)^{-1} = I_3 + D_4(\mathbf{v}^\varepsilon),$$

where $D_4(\mathbf{v}^\varepsilon)$ is given by

$$D_4(\mathbf{v}^\varepsilon) = \sum_{j=1}^{\infty} \left(\frac{2\varepsilon}{1-\varepsilon^2} \right)^j A(\mathbf{v}^\varepsilon)^j.$$

Here, $A(\mathbf{v}^\varepsilon) \mathbf{W} = \mathbf{v}^\varepsilon \times \mathbf{W}$ for a vector \mathbf{W} . We arrive at

$$\begin{aligned} \partial_s^4 \mathbf{v}^\varepsilon|_{s=0,1} &= \frac{1}{1-\varepsilon^2} (I_3 + D_4(\mathbf{v}^\varepsilon)) \{ -4(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon - 3|\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon - 6(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}_s^\varepsilon \\ &+ 2\varepsilon \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon + 4\varepsilon^2 (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + 2\varepsilon [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon + \varepsilon \mathbf{w}_2^\varepsilon \} |_{s=0,1}. \end{aligned}$$

Since,

$$\begin{aligned} D_4(\mathbf{v}^\varepsilon) &= \frac{2\varepsilon}{1-\varepsilon^2} \left(\sum_{j=0}^{\infty} \left(\frac{2\varepsilon}{1-\varepsilon^2} \right)^j A(\mathbf{v}^\varepsilon)^j \right) A(\mathbf{v}^\varepsilon) \\ &=: \frac{2\varepsilon}{1-\varepsilon^2} B_4(\mathbf{v}^\varepsilon) A(\mathbf{v}^\varepsilon), \end{aligned}$$

we have

$$\begin{aligned}
\partial_s^4 \mathbf{v}^\varepsilon|_{s=0,1} &= \frac{1}{1-\varepsilon^2} \left\{ -4(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon - 3|\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon - 6(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}_s^\varepsilon \right. \\
&\quad + 2\varepsilon \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon + 4\varepsilon^2 (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + 2\varepsilon [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon \Big\} \\
&\quad + \frac{2\varepsilon}{(1-\varepsilon^2)^2} B_4(\mathbf{v}^\varepsilon) \{ 2\varepsilon \mathbf{v}^\varepsilon \times (\mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) - 6(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon) \mathbf{v}_s^\varepsilon \} \\
&\quad + \frac{\varepsilon}{1-\varepsilon^2} (I_3 + D_4(\mathbf{v}^\varepsilon)) \mathbf{w}_2^\varepsilon|_{s=0,1}.
\end{aligned}$$

From $\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{ss}^\varepsilon|_{s=0,1} = -\varepsilon \mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon)|_{s=0,1}$ and (5.4), we finally have

$$\begin{aligned}
\partial_s^4 \mathbf{v}^\varepsilon|_{s=0,1} &= \frac{1}{1-\varepsilon^2} \left\{ -4(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon - 3|\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon + 2\varepsilon \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon \right. \\
&\quad + 4\varepsilon^2 (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + 2\varepsilon [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon \Big\} \\
&\quad + \frac{2\varepsilon}{(1-\varepsilon^2)^2} B_4(\mathbf{v}^\varepsilon) \{ 2\varepsilon \mathbf{v}^\varepsilon \times (\mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon) \} + \varepsilon \mathbf{w}_2^\varepsilon|_{s=0,1} \\
&= \frac{1}{1-\varepsilon^2} \left\{ -4(\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon - 3|\mathbf{v}_{ss}^\varepsilon|^2 \mathbf{v}^\varepsilon + 2\varepsilon \mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon \right. \\
&\quad + 4\varepsilon^2 (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + 2\varepsilon [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon \Big\} \\
&\quad + \frac{4\varepsilon^2}{(1-\varepsilon^2)^2} B_4(\mathbf{v}^\varepsilon) \{ [\mathbf{v}_s^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \mathbf{v}_{sss}^\varepsilon)] \mathbf{v}^\varepsilon \} + \varepsilon \mathbf{w}_2^\varepsilon|_{s=0,1},
\end{aligned}$$

where we have used the fact that

$$\|D_4(\mathbf{v}^\varepsilon)\|_{L^\infty(I \times [0,T])} + \|B_4(\mathbf{v}^\varepsilon)\|_{L^\infty(I \times [0,T])} \leq C$$

holds with $C > 0$ independent of ε because $|\mathbf{v}^\varepsilon| \equiv 1$. The above expression along with (5.4) and

$$\mathbf{v}_{tss}^\varepsilon|_{s=0,1} = \mathbf{v}^\varepsilon \times \partial_s^4 \mathbf{v}^\varepsilon + 2\mathbf{v}_s^\varepsilon \times \mathbf{v}_{sss}^\varepsilon + \varepsilon \partial_s^4 \mathbf{v}^\varepsilon + 2\varepsilon (\mathbf{v}_s^\varepsilon \cdot \mathbf{v}_{sss}^\varepsilon) \mathbf{v}^\varepsilon + \varepsilon \tilde{\mathbf{w}}_2^\varepsilon,$$

where $\tilde{\mathbf{w}}_2^\varepsilon$ are terms that satisfy the same form of estimate as \mathbf{w}_2^ε , yields the following estimate.

$$|[\mathbf{v}_{sss}^\varepsilon \cdot \mathbf{v}_{tss}^\varepsilon]|_{s=0,1}| \leq \frac{\varepsilon}{8} \|\mathbf{v}_{sss}^\varepsilon\|_1^2 + C \|\mathbf{v}_s^\varepsilon\|_2^6,$$

where we also utilized (5.3) and (5.4). The above estimate allows us to close the energy estimate as

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_{sss}^\varepsilon\|^2 \leq C \|\mathbf{v}_s^\varepsilon\|_s^2 (1 + \|\mathbf{v}_s^\varepsilon\|_2^6) - \frac{\varepsilon}{8} \|\partial_s^4 \mathbf{v}^\varepsilon\|^2.$$

Combining the estimates for \mathbf{v}_s^ε , $\mathbf{v}_{ss}^\varepsilon$, and $\mathbf{v}_{sss}^\varepsilon$ yields

$$\begin{aligned} \frac{d}{dt} \|\mathbf{v}_s^\varepsilon\|_2^2 &\leq C \|\mathbf{v}_s^\varepsilon\|_2^2 (1 + \|\mathbf{v}^\varepsilon\|_2^6) \\ &\leq C(1 + \|\mathbf{v}^\varepsilon\|_2^2)^4. \end{aligned}$$

Comparing $\|\mathbf{v}_s^\varepsilon\|_2^2$ with the solution of the ordinary differential equation given by

$$\begin{cases} R_t = C(1 + R)^4, & t > 0, \\ R(0) = \|\mathbf{v}_0^\varepsilon\|_2^2, \end{cases}$$

we see that there exists a $T_0 \in (0, T)$ depending on $\|\mathbf{v}_{0s}\|_2$ such that

$$\|\mathbf{v}_s^\varepsilon\|_2 \leq c_0,$$

where $c_0 > 0$ is a constant depending on $\|\mathbf{v}_{0s}\|_2$ and T_0 . Finally, we see that

$$\|\mathbf{v}_t^\varepsilon\| + \|\mathbf{v}_{ts}^\varepsilon\| \leq c_1,$$

holds from the equation of (3.1), where the dependence of c_1 is the same as c_0 . This finishes the proof of the proposition for $m = 1$.

We proceed by induction. Suppose the statement of the proposition is true up to $m - 1$ for some $m \geq 2$. From this assumption of induction, a solution $\mathbf{v}^\varepsilon \in Y_T^{m+2}(I)$ satisfies

$$\|\mathbf{v}^\varepsilon\|_{Y_{T_0}^{m-1}(I)} \leq c_*,$$

where $c_* > 0$ depends on $\|\mathbf{v}_0\|_{2(m-1)}$ and T_0 , and T_0 depends on $\|\mathbf{v}_{0s}\|_2$. Since the estimates for the t derivatives of \mathbf{v}^ε can be obtained from the equation, it is sufficient to prove the uniform estimate for $\partial_s^{2m} \mathbf{v}^\varepsilon$ and $\partial_s^{2m+1} \mathbf{v}^\varepsilon$. We have for $k_0 = 2m$ or $2m + 1$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_s^{k_0} \mathbf{v}^\varepsilon\|^2 &= (\partial_s^{k_0} \mathbf{v}^\varepsilon, \partial_s^{k_0} \mathbf{v}_t^\varepsilon) \\ &= (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon \times \partial_s^{k_0+2} \mathbf{v}^\varepsilon) + k_0 (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{v}_s^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon) \\ &\quad + \varepsilon (\partial_s^{k_0} \mathbf{v}^\varepsilon, \partial_s^{k_0+2} \mathbf{v}^\varepsilon) + k_0 \varepsilon (\partial_s^{k_0} \mathbf{v}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon) + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon) \\ &= (k_0 - 1) (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{v}_s^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon) - \varepsilon \|\partial_s^{k_0+1} \mathbf{v}^\varepsilon\|^2 \\ &\quad + k_0 \varepsilon (\partial_s^{k_0} \mathbf{v}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon) + [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon)]_{s=0}^1 \\ &\quad + \varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1 + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon), \end{aligned}$$

where $\mathbf{V}_{k_0}^\varepsilon$ are terms that can be estimated as

$$\|\mathbf{V}_{k_0}^\varepsilon\| \leq C \|\mathbf{v}_s^\varepsilon\|_{k_0-1},$$

with $C > 0$ depending on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$, and are harmless for the current estimate. From (5.3) and (5.4), we have

$$\begin{aligned}\mathbf{v}_s^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon &= -(\mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon + [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon] \mathbf{v}^\varepsilon \\ &= \frac{1}{2} \sum_{j=1}^{k_0} \binom{k_0+1}{j} (\partial_s^j \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1-j} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon + [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon] \mathbf{v}^\varepsilon,\end{aligned}$$

which gives

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\partial_s^{k_0} \mathbf{v}^\varepsilon\|^2 &= \frac{(k_0-1)}{2} \sum_{j=1}^{k_0} \binom{k_0+1}{j} (\partial_s^{k_0} \mathbf{v}^\varepsilon, (\partial_s^j \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1-j} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \\ &\quad + (k_0-1) (\partial_s^{k_0} \mathbf{v}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon] \mathbf{v}^\varepsilon) - \varepsilon \|\partial_s^{k_0+1} \mathbf{v}^\varepsilon\|^2 \\ &\quad + k_0 \varepsilon (\partial_s^{k_0} \mathbf{v}^\varepsilon, (\mathbf{v}_s^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon) + [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon)]_{s=0}^1 \\ &\quad + \varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1 + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon) \\ &\leq C \|\partial_s^{k_0} \mathbf{v}^\varepsilon\|^2 - \frac{\varepsilon}{2} \|\partial_s^{k_0+1} \mathbf{v}^\varepsilon\|^2 + (k_0-1) (\partial_s^{k_0} \mathbf{v}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon] \mathbf{v}^\varepsilon) \\ &\quad + [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon)]_{s=0}^1 + \varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1 + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon),\end{aligned}$$

where $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$. From (5.4), we further calculate

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\partial_s^{k_0} \mathbf{v}^\varepsilon\|^2 &\leq C \|\partial_s^{k_0} \mathbf{v}^\varepsilon\|^2 - \frac{\varepsilon}{2} \|\partial_s^{k_0+1} \mathbf{v}^\varepsilon\|^2 \\
&\quad - \frac{(k_0-1)}{2} \sum_{j=1}^{k_0-1} \binom{k_0}{j} (\partial_s^{k_0-j} \mathbf{v}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon] \partial_s^j \mathbf{v}^\varepsilon) \\
&\quad + [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon)]_{s=0}^1 + \varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1 + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon) \\
&= C \|\partial_s^{k_0} \mathbf{v}^\varepsilon\|^2 - \frac{\varepsilon}{2} \|\partial_s^{k_0+1} \mathbf{v}^\varepsilon\|^2 \\
&\quad + \frac{(k_0-1)}{2} \sum_{j=1}^{k_0-1} \binom{k_0}{j} (\partial_s^{k_0-j+1} \mathbf{v}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0} \mathbf{v}^\varepsilon] \partial_s^j \mathbf{v}^\varepsilon) \\
&\quad + \frac{(k_0-1)}{2} \sum_{j=1}^{k_0 s-1} \binom{k_0}{j} (\partial_s^{k_0-j} \mathbf{v}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_{ss}^\varepsilon) \cdot \partial_s^{k_0} \mathbf{v}^\varepsilon] \partial_s^j \mathbf{v}^\varepsilon) \\
&\quad + \frac{(k_0-1)}{2} \sum_{j=1}^{k_0-1} \binom{k_0}{j} (\partial_s^{k_0-j} \mathbf{v}^\varepsilon, [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0} \mathbf{v}^\varepsilon] \partial_s^{j+1} \mathbf{v}^\varepsilon) \\
&\quad - \frac{(k_0-1)}{2} \sum_{j=1}^{k_0-1} \binom{k_0}{j} [(\partial_s^{k_0-j} \mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon) [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0} \mathbf{v}^\varepsilon]]_{s=0}^1 \\
&\quad + [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon)]_{s=0}^1 + \varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1 + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon) \\
&\leq C \|\mathbf{v}_s^\varepsilon\|_{k_0-1}^2 - \frac{\varepsilon}{2} \|\partial_s^{k_0+1} \mathbf{v}^\varepsilon\|^2 \\
&\quad - \frac{(k_0-1)}{2} \sum_{j=1}^{k_0-1} \binom{k_0}{j} [(\partial_s^{k_0-j} \mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon) [(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{k_0} \mathbf{v}^\varepsilon]]_{s=0}^1 \\
&\quad + [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k_0+1} \mathbf{v}^\varepsilon)]_{s=0}^1 + \varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1 + (\partial_s^{k_0} \mathbf{v}^\varepsilon, \mathbf{V}_{k_0}^\varepsilon),
\end{aligned}$$

where $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$. We make use of Lemma 5.2 and 5.3 to estimate the boundary terms. First we focus on $\varepsilon [\partial_s^{k_0} \mathbf{v}^\varepsilon \cdot \partial_s^{k_0+1} \mathbf{v}^\varepsilon]_{s=0}^1$. Setting $s = 0, 1$ in (5.5) with m as $m+1$, we have

$$\begin{aligned}
\mathbf{0} &= \sum_{j=0}^{m+1} a_{m+1,j} \varepsilon^j A_{m+1-j} \partial_s^{2(m+1)} \mathbf{v}^\varepsilon + \sum_{j=0}^m \sum_{k=0}^{m-j} e_{m+1,j,k} \varepsilon^j A_{m-j-k} \mathbf{v}_s^\varepsilon \times (A_k \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon) \\
&\quad + \sum_{j=1}^{m+1} b_{m+1,j} \varepsilon^{m+2-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon \Big|_{s=0,1}.
\end{aligned}$$

As we showed in Section 3, for $i \geq 1$ we have

$$A_i \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon = \begin{cases} (-1)^n \mathbf{v}^\varepsilon \times \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon, & i = 2n + 1, \\ (-1)^{n+1} ((\mathbf{v}^\varepsilon \cdot \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon), & i = 2n, \end{cases}$$

which gives

$$\mathbf{v}_s^\varepsilon \times (A_i \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon) = \begin{cases} (-1)^n (\mathbf{v}_s^\varepsilon \cdot \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon, & i = 2n + 1, \\ (-1)^{n+1} \{ (\mathbf{v}^\varepsilon \cdot \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon) \mathbf{v}_s^\varepsilon \times \mathbf{v}^\varepsilon \\ - \mathbf{v}_s^\varepsilon \times \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon \}, & i = 2n. \end{cases}$$

Finally, if we take the exterior product of the above with \mathbf{v}^ε from the left, we see that $\mathbf{v}^\varepsilon \times (\mathbf{v}_s^\varepsilon \times (A_i \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon))$ is identically zero. Hence we have shown that

$$A_k \mathbf{v}_s^\varepsilon \times (A_i \partial_s^{2(m+1)-1} \mathbf{v}^\varepsilon) = \mathbf{0}$$

if $k, i \geq 1$. Hence we have

$$\begin{aligned} \mathbf{0} &= \sum_{j=0}^{m+1} a_{m+1,j} \varepsilon^j A_{m+1-j} \partial_s^{2m+2} \mathbf{v}^\varepsilon \\ &+ \sum_{j=0}^m \varepsilon^j \{ e_{m+1,j,0} A_{m-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + e_{m+1,j,m-j} \mathbf{v}_s^\varepsilon \times (A_{m-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon) \} \\ &+ \sum_{j=1}^{m+1} b_{m+1,j} \varepsilon^{m+2-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon|_{s=0,1}. \end{aligned}$$

Suppose $m+1$ is even and set $m+1 = 2n$. The first term on the right-hand side can be calculated as

$$\begin{aligned} \sum_{j=0}^{m+1} a_{m+1,j} \varepsilon^j A_{m+1-j} \partial_s^{2m+2} \mathbf{v}^\varepsilon &= \sum_{k=0}^n a_{m+1,2k} \varepsilon^{2k} (-1)^{k+1} ((\mathbf{v}^\varepsilon \cdot \partial_s^{2m+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon - \partial_s^{2m+2} \mathbf{v}^\varepsilon) \\ &+ \sum_{k=0}^{n-1} a_{m+1,2k+1} \varepsilon^{2k+1} (-1)^k \mathbf{v}^\varepsilon \times \partial_s^{2m+2} \mathbf{v}^\varepsilon \\ &= \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k \right) \partial_s^{2m+2} \mathbf{v}^\varepsilon \\ &+ \left(\sum_{k=0}^{n-1} a_{m+1,2k+1} \varepsilon^{2k+1} (-1)^k \right) \mathbf{v}^\varepsilon \times \partial_s^{2m+2} \mathbf{v}^\varepsilon \\ &+ \sum_{k=0}^n a_{m+1,2k} \varepsilon^{2k} (-1)^{k+1} (\mathbf{v}^\varepsilon \cdot \partial_s^{2m+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon, \end{aligned}$$

where we used $a_{m+1,0} = 1$. Hence we have

$$\begin{aligned}
& \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k\right) \partial_s^{2m+2} \mathbf{v}^\varepsilon + \left(\sum_{k=0}^{n-1} a_{m+1,2k+1} \varepsilon^{2k+1} (-1)^k\right) \mathbf{v}^\varepsilon \times \partial_s^{2m+2} \mathbf{v}^\varepsilon \Big|_{s=0,1} \\
&= - \sum_{k=0}^n a_{m+1,2k} \varepsilon^{2k} (-1)^{k+1} (\mathbf{v}^\varepsilon \cdot \partial_s^{2m+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \\
&\quad - \sum_{j=0}^m \varepsilon^j \{ e_{m+1,j,0} A_{m-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + e_{m+1,j,m-j} \mathbf{v}_s^\varepsilon \times (A_{m-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon) \} \\
&\quad - \sum_{j=1}^{m+1} b_{m+1,j} \varepsilon^{m+2-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon \Big|_{s=0,1}.
\end{aligned}$$

Then there exists a $\varepsilon_{m+1} \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_{m+1})$, the matrix

$$\left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k\right) I_3 - \varepsilon \left(\sum_{k=0}^{n-1} a_{m+1,2k+1} \varepsilon^{2k} (-1)^{k+1}\right) A(\mathbf{v}^\varepsilon)$$

is reversible and the inverse can be expressed as

$$\begin{aligned}
& \left\{ \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k\right) I_3 - \varepsilon \left(\sum_{k=0}^{n-1} a_{m+1,2k+1} \varepsilon^{2k} (-1)^{k+1}\right) A(\mathbf{v}^\varepsilon) \right\}^{-1} \\
&= \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k\right) (I_3 + D_{m+1}(\mathbf{v}^\varepsilon)),
\end{aligned}$$

where

$$D_{m+1}(\mathbf{v}^\varepsilon) = \sum_{j=1}^{\infty} \varepsilon^j M^j A(\mathbf{v}^\varepsilon)^j,$$

with

$$M = \left(\sum_{k=0}^{n-1} a_{m+1,2k+1} \varepsilon^{2k} (-1)^{k+1}\right) \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k\right)^{-1}.$$

Hence we have

$$\begin{aligned}
\partial_s^{2m+2} \mathbf{v}^\varepsilon|_{s=0,1} &= \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k\right) (I_3 + D_{m+1}(\mathbf{v}^\varepsilon)) \\
&\quad \left\{ - \sum_{k=0}^n a_{m+1,2k} \varepsilon^{2k} (-1)^{k+1} (\mathbf{v}^\varepsilon \cdot \partial_s^{2m+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \right. \\
&\quad - \sum_{j=0}^m \varepsilon^j \{ e_{m+1,j,0} A_{m-j}(\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + e_{m+1,j,m-j} \mathbf{v}_s^\varepsilon \times (A_{m-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon) \} \\
&\quad \left. - \sum_{j=1}^{m+1} b_{m+1,j} \varepsilon^{m+2-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon \right\} \Big|_{s=0,1}.
\end{aligned}$$

Since

$$\begin{aligned}
D_{m+1}(\mathbf{v}^\varepsilon) &= \varepsilon \left(\sum_{j=0}^{\infty} \varepsilon^j M^{j+1} A(\mathbf{v}^\varepsilon)^j \right) A(\mathbf{v}^\varepsilon) \\
&=: \varepsilon B_{m+1}(\mathbf{v}^\varepsilon) A(\mathbf{v}^\varepsilon)
\end{aligned}$$

we have

$$\begin{aligned}
(5.11) \quad \partial_s^{2m+2} \mathbf{v}^\varepsilon|_{s=0,1} &= C_{m+1} \left\{ - \sum_{k=0}^n a_{m+1,2k} \varepsilon^{2k} (-1)^{k+1} (\mathbf{v}^\varepsilon \cdot \partial_s^{2m+2} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \right. \\
&\quad - \sum_{j=0}^m \varepsilon^j \{ e_{m+1,j,0} A_{m-j}(\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) \\
&\quad \quad + e_{m+1,j,m-j} \mathbf{v}_s^\varepsilon \times (A_{m-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon) \} \\
&\quad \left. - \sum_{j=1}^{m+1} b_{m+1,j} \varepsilon^{m+2-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon \right\} \\
&\quad + C_{m+1} B_{m+1}(\mathbf{v}^\varepsilon) \left\{ - \sum_{j=0}^m \varepsilon^j \{ e_{m+1,j,0} A_{m+1-j}(\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) \} \right. \\
&\quad \left. + \varepsilon (C_{m+1} I_3 + \varepsilon B_{m+1}(\mathbf{v}^\varepsilon)) \mathbf{U}_{m+1}^\varepsilon \right\} \Big|_{s=0,1},
\end{aligned}$$

where we have set

$$C_{m+1} = \left(1 + \sum_{k=1}^n a_{m+1,2k} \varepsilon^{2k} (-1)^k \right)^{-1},$$

Lastly, since

$$A_k(\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) = \begin{cases} (-1)^{l+1}(\mathbf{v}^\varepsilon \cdot \partial_s^{2m+1} \mathbf{v}^\varepsilon) \mathbf{v}_s^\varepsilon, & k = 2l - 1, \\ (-1)^{l+1}(\mathbf{v}^\varepsilon \cdot \partial_s^{2m+1} \mathbf{v}^\varepsilon) \mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon, & k = 2l, \end{cases}$$

This shows that combining (5.11) and (5.4), we have the estimate

$$\varepsilon \left| [\partial_s^{2m+1} \mathbf{v}^\varepsilon \cdot \partial_s^{2m+2} \mathbf{v}^\varepsilon]_{s=0}^1 \right| \leq \frac{\varepsilon}{16} \|\partial_s^{2m+2} \mathbf{v}^\varepsilon\|^2 + C \|\mathbf{v}_s^\varepsilon\|_{2m}^2,$$

holds, where $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$ and we also utilized the fact that

$$\|B_{m+1}(\mathbf{v}^\varepsilon)\|_{L^\infty(I \times [0, T])} \leq C$$

with $C > 0$ independent of ε .

Next, we consider $[\partial_s^k \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{k+1} \mathbf{v}^\varepsilon)]_{s=0}^1$. We further calculate the right-hand side of (5.6). By Lemma 5.2 with $m = l$ for $2 \leq l \leq m$, we have

$$\begin{aligned} \partial_s^{2(m+1-l)} \partial_t^l \mathbf{v}^\varepsilon &= \sum_{j=0}^l a_{l,j} \varepsilon^j A_{l-j} \partial_s^{2m+2} \mathbf{v}^\varepsilon + 2m \sum_{j=0}^{l-1} a_{l,j} \varepsilon^j \sum_{i=0}^{l-1-j} A_{l-1-j-i} \mathbf{v}_s^\varepsilon \times (A_i \partial_s^{2m+1} \mathbf{v}^\varepsilon) \\ &\quad + \sum_{j=0}^{l-1} \sum_{k=0}^{l-1-j} e_{l,j,k} \varepsilon^j A_{l-1-j-k} \mathbf{v}_s^\varepsilon \times (A_k \partial_s^{2m+1} \mathbf{v}^\varepsilon) \\ &\quad + \sum_{j=1}^l b_{l,j} \varepsilon^{l+1-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon \\ \partial_t^m \mathbf{v}_s^\varepsilon &= \sum_{j=0}^m a_{m,j} \varepsilon^j A_{m-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon, \end{aligned}$$

where as in the proof of Lemma 5.6, \mathbf{U}_m^ε denotes the collection of terms that satisfy the estimate stated in Lemma 5.5, and may change from line to line. This yields

$$\begin{aligned} \partial_s^{2(m+1-l)} \partial_t^l \mathbf{v}^\varepsilon &= \sum_{j=0}^l a_{l,j} \varepsilon^j A_{l-j} \partial_s^{2m+2} \mathbf{v}^\varepsilon \\ &\quad + 2m \sum_{j=0}^{l-1} a_{l,j} \varepsilon^j (A_{l-1-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + \mathbf{v}_s^\varepsilon \times (A_{l-1-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon)) \\ &\quad + \sum_{j=0}^{l-1} \varepsilon^j (e_{l,j,0} A_{l-1-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + e_{l,j,l-1-j} \mathbf{v}_s^\varepsilon \times (A_{l-1-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon)) \\ &\quad + \sum_{j=1}^l b_{l,j} \varepsilon^{l+1-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon + \mathbf{U}_{m+1}^\varepsilon. \end{aligned}$$

Substituting these expressions for $\partial_s^{2(m+1-l)} \partial_t^l \mathbf{v}^\varepsilon$ and $\partial_t^m \mathbf{v}_s^\varepsilon$ into the right-hand side of (5.6) with m as $m+1$ yields

(5.12)

$$\begin{aligned}
\mathbf{v}^\varepsilon \times \partial_s^{2m+2} \mathbf{v}^\varepsilon|_{s=0,1} &= (-1)^{m+1} \varepsilon A_m \left\{ \sum_{j=0}^m a_{m,j} \varepsilon^j \partial_s^{2m+2} \mathbf{v}^\varepsilon \right. \\
&+ 2m \sum_{j=0}^{m-1} a_{m,j} \varepsilon^j (A_{m-1-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + \mathbf{v}_s^\varepsilon \times (A_{m-1-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon)) \\
&+ \sum_{j=0}^{m-1} \varepsilon^j \{ e_{m,j,0} A_{m-1-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) \\
&\quad + e_{m,j,m-1-j} \mathbf{v}_s^\varepsilon \times (A_{m-1-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon) \} \\
&+ \sum_{j=1}^m b_{m,j} \varepsilon^{m+1-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon \\
&+ 2(\mathbf{v}_s^\varepsilon \cdot [\sum_{j=0}^m \varepsilon^j A_{m-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon]) \mathbf{v}^\varepsilon \Big\} \\
&+ \varepsilon \left\{ \sum_{l=0}^{m-1} (-1)^l A_{l+2} \left[\sum_{j=0}^l a_{l,j} \varepsilon^j A_{l-j} \partial_s^{2m+2} \mathbf{v}^\varepsilon \right. \right. \\
&+ 2m \sum_{j=0}^{l-1} a_{l,j} \varepsilon^j (A_{l-1-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) + \mathbf{v}_s^\varepsilon \times (A_{l-1-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon)) \\
&+ \sum_{j=0}^{l-1} \varepsilon^j [e_{l,j,0} A_{l-1-j} (\mathbf{v}_s^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon) \\
&\quad + e_{l,j,l-1-j} \mathbf{v}_s^\varepsilon \times (A_{l-1-j} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \\
&\left. \left. + \sum_{j=1}^l b_{l,j} \varepsilon^{l+1-j} [\mathbf{v}_s^\varepsilon \cdot (A_{j-1} \partial_s^{2m+1} \mathbf{v}^\varepsilon)] \mathbf{v}^\varepsilon \right] \right\} + \varepsilon \mathbf{U}_{m+1}^\varepsilon|_{s=0,1}.
\end{aligned}$$

Substituting (5.11) in the right-hand side, re-writing terms of the form $A_k \mathbf{W}$ according

to k , and utilizing (5.4) and (5.7), we can see that the following estimates hold.

$$\begin{aligned} |[\partial_s^{2m+1} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{2m+2} \mathbf{v}^\varepsilon)]_{s=0}^1| &\leq \frac{\varepsilon}{16} \|\partial_s^{2m+2} \mathbf{v}^\varepsilon\|^2 + C \|\mathbf{v}_s^\varepsilon\|_{2m}^2 \\ |[\partial_s^{2m} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{2m+1} \mathbf{v}^\varepsilon)]_{s=0}^1| &= |[\partial_s^{2m+1} \mathbf{v}^\varepsilon \cdot (\mathbf{v}^\varepsilon \times \partial_s^{2m} \mathbf{v}^\varepsilon)]_{s=0}^1| \\ &\leq \frac{\varepsilon}{16} \|\partial_s^{2m+2} \mathbf{v}^\varepsilon\|^2 + C \|\mathbf{v}_s^\varepsilon\|_{2m}^2 \end{aligned}$$

where $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$, but not on ε . We also mention that the above estimate holds because every term on the right-hand side of (5.12) contains ε . Finally from (5.7), we have

$$\begin{aligned} |[(\partial_s^{2m+1-j} \mathbf{v}^\varepsilon \cdot \partial_s^j \mathbf{v}^\varepsilon)[(\mathbf{v}^\varepsilon \times \mathbf{v}_s^\varepsilon) \cdot \partial_s^{2m+1} \mathbf{v}^\varepsilon]]_{s=0}^1| &\leq 2\varepsilon |Y_{2m+1-j,j}| \|\mathbf{v}_s^\varepsilon\| \|\partial_s^{2m+1} \mathbf{v}^\varepsilon\|_{s=0,1} \\ &\leq \frac{\varepsilon}{16} \|\partial_s^{2m+2} \mathbf{v}^\varepsilon\|^2 + C \|\mathbf{v}_s^\varepsilon\|_{2m}^2. \end{aligned}$$

Combining all of the estimates obtained for the boundary terms with the estimate we have derived so far for $\partial_s^{k_0} \mathbf{v}^\varepsilon$ yields

$$\frac{1}{2} \frac{d}{dt} \|\partial_s^{2m} \mathbf{v}^\varepsilon\|_1^2 \leq C \|\mathbf{v}_s^\varepsilon\|_{2m}^2 - \frac{\varepsilon}{8} \|\partial_s^{2m+1} \mathbf{v}^\varepsilon\|_1^2,$$

where $C > 0$ depends on $\|\mathbf{v}_s^\varepsilon\|_{2(m-1)}$. If we set $\varepsilon_0 := \min\{\varepsilon_l \mid 1 \leq l \leq m+1\}$, then from Gronwall's inequality and the assumption of induction, we see that there exists $c_* > 0$ depending on $\|\mathbf{v}_{0s}\|_{2m}$ and T_0 , and T_0 depending on $\|\mathbf{v}_{0s}\|_2$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\sup_{0 \leq t \leq T_0} \|\mathbf{v}_s^\varepsilon(t)\|_{2m} \leq c_*.$$

Estimating the t derivatives of \mathbf{v}^ε via the equation in (3.1), we obtain the necessary estimates of \mathbf{v}^ε in $Y_{T_0}^m(I)$ and this finishes the proof of Proposition 5.4. \square

5.2 Taking the limit $\varepsilon \rightarrow +0$

Now we take the limit $\varepsilon \rightarrow +0$. Fix $m \geq 1$ and for $\varepsilon \in (0, \varepsilon_0]$, let $\mathbf{v}^\varepsilon \in Y_{T_0}^{m+2}(I)$ be the solution of (3.1) satisfying the uniform estimate given in Proposition 5.4. Let $\varepsilon, \varepsilon' \in (0, \varepsilon_0]$, $\varepsilon < \varepsilon'$, and set $\mathbf{Z} := \mathbf{v}^{\varepsilon'} - \mathbf{v}^\varepsilon$. We see that \mathbf{Z} satisfies

$$\begin{cases} \mathbf{Z}_t = \mathbf{v}^\varepsilon \times \mathbf{Z}_{ss} + \mathbf{Z} \times \mathbf{v}_{ss}^{\varepsilon'} + \varepsilon' \mathbf{Z}_{ss} + (\varepsilon' - \varepsilon) \mathbf{v}_{ss}^\varepsilon + \varepsilon' |\mathbf{v}_s^{\varepsilon'}|^2 \mathbf{v}^{\varepsilon'} - \varepsilon |\mathbf{v}_s^\varepsilon|^2 \mathbf{v}^\varepsilon, & s > 0, t > 0, \\ \mathbf{Z}(s, 0) = \mathbf{0}, & s > 0, \\ \mathbf{Z}(0, t) = \mathbf{Z}(1, t) = \mathbf{0}, & t > 0. \end{cases}$$

We have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{Z}\|^2 &= (\mathbf{Z}, \mathbf{Z}_t) \leq (\mathbf{Z}, \mathbf{v}^\varepsilon \times \mathbf{Z}_{ss}) + \varepsilon' (\mathbf{Z}, \mathbf{Z}_{ss}) + (\varepsilon' + \varepsilon) c_* \\ &\leq c_* \|\mathbf{Z}\|_1^2 - \varepsilon' \|\mathbf{Z}_s\|^2 + (\varepsilon' + \varepsilon) c_* \end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathbf{Z}_s\|^2 &= (\mathbf{Z}_s, \mathbf{Z}_{ts}) = -(\mathbf{Z}_{ss}, \mathbf{Z}_t) \\
&= -(\mathbf{Z}_{ss}, \mathbf{Z} \times \mathbf{v}_{ss}^{\varepsilon'}) - \varepsilon' \|\mathbf{Z}_{ss}\|^2 - (\varepsilon' - \varepsilon)(\mathbf{Z}_{ss}, \mathbf{v}_{ss}^{\varepsilon}) \\
&\quad - \varepsilon' (\mathbf{Z}_{ss}, |\mathbf{v}_s^{\varepsilon'}| \mathbf{v}^{\varepsilon'}) + \varepsilon (\mathbf{Z}_{ss}, |\mathbf{v}_s^{\varepsilon}|^2 \mathbf{v}^{\varepsilon}) \\
&\leq c_* \|\mathbf{Z}\|_1^2 - \frac{\varepsilon'}{2} \|\mathbf{Z}_{ss}\|^2 + c_*(\varepsilon' + \varepsilon),
\end{aligned}$$

where $c_* > 0$ depends on the uniform estimate of \mathbf{v}^{ε} in $Y_{T_0}^1(I)$. Hence by Gronwall's inequality, we see that

$$\sup_{0 \leq t \leq T_0} \|\mathbf{Z}(t)\|_1^2 \leq c_* e^{c_* T_0} (\varepsilon' + \varepsilon),$$

which proves that $\mathbf{Z} \rightarrow \mathbf{0}$ in $C([0, T_0]; H^1(I))$. Hence there exists $\mathbf{v} \in C([0, T_0]; H^1(I))$ such that $\mathbf{v}^{\varepsilon} \rightarrow \mathbf{v}$. The uniform estimate and the interpolation inequality implies that $\mathbf{v}^{\varepsilon} \rightarrow \mathbf{v}$ in $\bigcap_{j=0}^m C^j([0, T_0]; H^{2(m-j)}(I))$, and this \mathbf{v} is the desired solution of (1.5). Note that from the uniform estimate, $\partial_t^j \mathbf{v} \in \bigcap_{j=0}^m L^\infty(0, T_0; H^{2(m-j)+1}(I))$ can be shown from an argument utilizing weak* convergence, but this does not benefit us because the energy estimate for the solution of the original problem (1.5) is derived in Sobolev spaces with different indices, which will be shown in the next subsection.

5.3 Energy estimate of the solution to (1.5)

Up until now, we have assumed that the solution is smooth as we need it to be, which is possible because the initial datum can be approximated by smooth ones from Proposition 3.6 and 3.7. We derive an a priori estimate of the solution to (1.5) to justify the approximation argument. We emphasize this point because the estimate obtained here and the uniform estimate obtained in the last subsection are derived in Sobolev spaces with different indices. Fix an arbitrary non-negative integer l . We derive the estimate in the class $H^{l+3}(I)$. For any $T > 0$, let $\mathbf{v} \in \bigcap_{j=0}^{m+2} C^j([0, T]; H^{2(m+2-j)}(I))$ be the solution of (1.5) obtained in the last subsection such that $l + 5 \leq 2m + 2$. We see that

$$\frac{1}{2} \frac{d}{dt} |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}_t \equiv 0,$$

which shows that $|\mathbf{v}| \equiv 1$. Hence we can prove that (5.3) and (5.4) also holds even if we replace \mathbf{v}^{ε} with \mathbf{v} . Next, we estimate conserved quantities which were utilized in the

analysis of the Cauchy problem in Nishiyama and Tani [9].

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_s\|^2 &= -(\mathbf{v}_{ss}, \mathbf{v}_t) + [\mathbf{v}_s \cdot \mathbf{v}_t]_{s=0}^1 = 0 \\
\frac{d}{dt} \left\{ \|\mathbf{v}_{ss}\|^2 - \frac{5}{4} \|\mathbf{v}_s\|^2 \right\} &= 2(\mathbf{v}_{ss}, \mathbf{v}_{tss}) - 5 \int_0^1 |\mathbf{v}_s|^2 \mathbf{v}_s \cdot \mathbf{v}_{ts} ds \\
&= -2(\mathbf{v}_{sss}, \mathbf{v}_{ts}) - 5 \int_0^1 |\mathbf{v}_s|^2 \mathbf{v}_s \cdot \mathbf{v}_{ts} ds \\
&\quad + 2[\mathbf{v}_{ss} \cdot \mathbf{v}_{ts}]_{s=0}^1 \\
&= \left[-3|\mathbf{v}_s|^2 \mathbf{v}_{ss} \cdot (\mathbf{v} \times \mathbf{v}_s) \right]_{s=0}^1 + 2[\mathbf{v}_{ss} \cdot \mathbf{v}_{ts}]_{s=0}^1 \\
&= \left[3|\mathbf{v}_s|^2 \mathbf{v}_s \cdot (\mathbf{v} \times \mathbf{v}_{ss}) \right]_{s=0}^1 + 2[\mathbf{v}_{ss} \cdot \mathbf{v}_{ts}]_{s=0}^1
\end{aligned}$$

From $\mathbf{v}_t|_{s=0,1} = \mathbf{0}$, we have

$$\mathbf{v} \times \mathbf{v}_{ss}|_{s=0,1} = \mathbf{0},$$

which shows that the first boundary term is zero. Direct calculation also yields

$$\begin{aligned}
\mathbf{v}_{ss} \cdot \mathbf{v}_{ts}|_{s=0,1} &= \mathbf{v}_{ss} \cdot (\mathbf{v} \times \mathbf{v}_{sss} + \mathbf{v}_s \times \mathbf{v}_{ss})|_{s=0,1} = \mathbf{v}_{ss} \cdot (\mathbf{v} \times \mathbf{v}_{sss}) \\
&= -\mathbf{v}_{sss} \cdot (\mathbf{v} \times \mathbf{v}_{ss})|_{s=0,1} \\
&= \mathbf{0},
\end{aligned}$$

which proves

$$\frac{d}{dt} \left\{ \|\mathbf{v}_{ss}\|^2 - \frac{5}{4} \|\mathbf{v}_s\|^2 \right\} = 0.$$

Similarly, we can prove that

$$\begin{aligned}
&\frac{d}{dt} \left\{ \|\mathbf{v}_{sss}\|^2 - \frac{7}{2} \|\mathbf{v}_s\| \|\mathbf{v}_{ss}\|^2 - 14 \|\mathbf{v}_s \cdot \mathbf{v}_{ss}\|^2 + \frac{21}{8} \|\mathbf{v}_s\|^3 \right\} \\
&= 2[\mathbf{v}_{sss} \cdot \mathbf{v}_{tss}]_{s=0}^1 \\
&\quad + \left[18(\mathbf{v}_s \cdot \mathbf{v}_{ss}) \mathbf{v}_{sss} \cdot (\mathbf{v} \times \mathbf{v}_s) + 5|\mathbf{v}_s|^2 \mathbf{v}_{sss} \cdot (\mathbf{v} \times \mathbf{v}_{ss}) \right. \\
&\quad \left. - 2(\mathbf{v}_s \cdot \mathbf{v}_{sss})(\mathbf{v} \times \mathbf{v}_s) \cdot \mathbf{v}_{ss} + |\mathbf{v}_{ss}|^2 \mathbf{v}_{ss} \cdot (\mathbf{v} \times \mathbf{v}_s) \right. \\
&\quad \left. - \frac{27}{4} |\mathbf{v}_s|^2 (\mathbf{v} \times \mathbf{v}_s) \cdot \mathbf{v}_{ss} \right]_{s=0}^1 \\
&= 0.
\end{aligned}$$

From these estimates and the interpolation inequality, we see that

$$\sup_{0 \leq t \leq T} \|\mathbf{v}_s(t)\|_2 \leq C \|\mathbf{v}_{0s}\|_2^2 (1 + \|\mathbf{v}_{0s}\|_2^2)^4,$$

where $C > 0$ is monotone increasing with respect to $T > 0$. We continue by induction. Suppose that \mathbf{v} satisfies

$$\sup_{0 \leq t \leq T} \|\mathbf{v}_s(t)\|_{k-1} \leq C_*,$$

where $C_* > 0$ depends on $\|\mathbf{v}_{0s}\|_{k-1}$ and is monotone increasing in $T > 0$ for some $k \geq 3$. We calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_s^{k+1} \mathbf{v}\|^2 &= (\partial_s^{k+1} \mathbf{v}, \partial_s^{k+1} \mathbf{v}_t) = (\partial_s^{k+1} \mathbf{v}, \mathbf{v} \times \partial_s^{k+3} \mathbf{v}) + k(\partial_s^{k+1} \mathbf{v}, \mathbf{v}_s \times \partial_s^{k+2} \mathbf{v}) \\ &\quad + (\partial_s^{k+1} \mathbf{v}, \mathbf{h}_{k+1}), \end{aligned}$$

where \mathbf{h}_{k+1} are terms that satisfy

$$\|\mathbf{h}_{k+1}\| \leq C \|\mathbf{v}_s\|_k,$$

with $C > 0$ depending on C_* . We further calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_s^{k+1} \mathbf{v}\|^2 &= -k(\partial_s^{k+1} \mathbf{v}, \mathbf{v}_s \times \partial_s^{k+2} \mathbf{v}) + [\partial_s^{k+1} \mathbf{v} \cdot (\mathbf{v} \times \partial_s^{k+2} \mathbf{v})]_{s=0}^1 + (\partial_s^{k+1} \mathbf{v}, \mathbf{h}_{k+1}) \\ &= -k(\partial_s^{k+1} \mathbf{v}, (\mathbf{v} \cdot \partial_s^{k+2} \mathbf{v}) \mathbf{v} \times \mathbf{v}_s) + k(\partial_s^{k+1} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+2} \mathbf{v}] \mathbf{v}) \\ &\quad + [\partial_s^{k+1} \mathbf{v} \cdot (\mathbf{v} \times \partial_s^{k+2} \mathbf{v})]_{s=0}^1 + (\partial_s^{k+1} \mathbf{v}, \mathbf{h}_{k+1}), \end{aligned}$$

where (5.3) was used in the last equality. Furthermore, from (5.4) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_s^{k+1} \mathbf{v}\|^2 &= \frac{k}{2} \sum_{j=1}^{k+1} \binom{k+2}{j} (\partial_s^{k+1} \mathbf{v}, (\partial_s^j \mathbf{v} \cdot \partial_s^{k+2-j} \mathbf{v}) \mathbf{v} \times \mathbf{v}_s) \\ &\quad - \frac{k}{2} \sum_{j=1}^k \binom{k+1}{j} (\partial_s^{k+1-j} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+2} \mathbf{v}] \partial_s^j \mathbf{v}) \\ &\quad + [\partial_s^{k+1} \mathbf{v} \cdot (\mathbf{v} \times \partial_s^{k+2} \mathbf{v})]_{s=0}^1 + (\partial_s^{k+1} \mathbf{v}, \mathbf{h}_{k+1}) \\ &\leq C \|\mathbf{v}_s\|_k^2 - \frac{k}{2} \sum_{j=1}^k \binom{k+1}{j} (\partial_s^{k+1-j} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+2} \mathbf{v}] \partial_s^j \mathbf{v}) \\ &\quad + [\partial_s^{k+1} \mathbf{v} \cdot (\mathbf{v} \times \partial_s^{k+2} \mathbf{v})]_{s=0}^1, \end{aligned}$$

where $C > 0$ depends on C_* . Taking the limit $\varepsilon \rightarrow +0$ in Lemma 5.6, we see that \mathbf{v} satisfies

$$(5.13) \quad \mathbf{v} \times \partial_s^{2m} \mathbf{v}|_{s=0,1} = \mathbf{0},$$

$$(5.14) \quad \partial_s^i \mathbf{v} \cdot \partial_s^n \mathbf{v}|_{s=0,1} = 0 \quad (i + n = 2m + 1),$$

for $m \geq 1$. Hence, the boundary term $[\partial_s^{k+1} \mathbf{v} \cdot (\mathbf{v} \times \partial_s^{k+2} \mathbf{v})]_{s=0}^1$ is zero from (5.13) because either $k + 1$ or $k + 2$ is even. This shows that

$$\frac{1}{2} \frac{d}{dt} \|\partial_s^{k+1} \mathbf{v}\|^2 \leq C \|\mathbf{v}_s\|_k^2 - \frac{k}{2} \sum_{j=1}^k \binom{k+1}{j} (\partial_s^{k+1-j} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+2} \mathbf{v}] \partial_s^j \mathbf{v}).$$

We continue as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_s^{k+1} \mathbf{v}\|^2 &\leq C \|\mathbf{v}_s\|_k^2 + \frac{k}{2} \sum_{j=1}^k \binom{k+1}{j} \left\{ (\partial_s^{k+j} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+1} \mathbf{v}] \partial_s^j \mathbf{v}) \right. \\ &\quad + (\partial_s^{k+1-j} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_{ss}) \cdot \partial_s^{k+1} \mathbf{v}] \partial_s^j \mathbf{v}) + (\partial_s^{k+1-j} \mathbf{v}, [(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+1} \mathbf{v}] \partial_s^{j+1} \mathbf{v}) \\ &\quad \left. - [(\partial_s^{k+1-j} \mathbf{v} \cdot \partial_s^j \mathbf{v}) ((\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+1} \mathbf{v})]_{s=0}^1 \right\} \\ &\leq C \|\mathbf{v}_s\|_k^2 - \frac{k}{2} \sum_{j=1}^k \binom{k+1}{j} [(\partial_s^{k+1-j} \mathbf{v} \cdot \partial_s^j \mathbf{v}) ((\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{k+1} \mathbf{v})]_{s=0}^1. \end{aligned}$$

When $k = 2m - 1$, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_s^{2m} \mathbf{v}\|^2 \\ &\leq C \|\mathbf{v}_s\|_{2m-1}^2 - \frac{(2m-1)}{2} \sum_{j=1}^{2m-2} \binom{2m}{j} [(\partial_s^{2m-j} \mathbf{v} \cdot \partial_s^j \mathbf{v}) ((\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{2m} \mathbf{v})]_{s=0}^1. \end{aligned}$$

From (5.13), we see that

$$(\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{2m} \mathbf{v}|_{s=0,1} = -(\mathbf{v} \times \partial_s^{2m} \mathbf{v}) \cdot \mathbf{v}_s|_{s=0,1} = 0,$$

and we have

$$\frac{1}{2} \frac{d}{dt} \|\partial_s^{2m} \mathbf{v}\|^2 \leq C \|\mathbf{v}_s\|_{2m-1}^2.$$

When $k = 2m$, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_s^{2m+1} \mathbf{v}\|^2 \\ &\leq C \|\mathbf{v}_s\|^2 - m \sum_{j=1}^{2m} \binom{2m+1}{j} [(\partial_s^{2m+1-j} \mathbf{v} \cdot \partial_s^j \mathbf{v}) ((\mathbf{v} \times \mathbf{v}_s) \cdot \partial_s^{2m+1} \mathbf{v})]_{s=0}^1. \end{aligned}$$

We see from (5.14) that

$$\partial_s^{2m+1-j} \mathbf{v} \cdot \partial_s^j \mathbf{v}|_{s=0,1} = 0,$$

for all j with $1 \leq j \leq 2m$, yielding

$$\frac{1}{2} \frac{d}{dt} \|\partial_s^{2m+1} \mathbf{v}\|^2 \leq C \|\mathbf{v}_s\|_{2m}^2.$$

In either cases, we have

$$\frac{1}{2} \frac{d}{dt} \|\partial_s^{k+1} \mathbf{v}\|^2 \leq C \|\mathbf{v}_s\|_k^2,$$

which, along with the assumption of induction and Gronwall's inequality, allows us to conclude that

$$\sup_{0 \leq t \leq T} \|\mathbf{v}_s\|_k \leq C_*,$$

where $C_* > 0$ depends on $\|\mathbf{v}_{0s}\|_k$ and is monotone increasing in $T > 0$. The t derivatives can be estimated from the equation in (1.5) and thus, we have derived an a priori estimate in the function space stated in Theorem 2.2.

Combining Proposition 3.6 and 3.7, the time-local existence of a smooth solution to (1.5), and the a priori estimate just obtained, we can conclude that by a standard approximation and continuation argument, for an arbitrary $T > 0$ we have a solution \mathbf{v} of (1.5) satisfying

$$\mathbf{v} \in \bigcap_{j=0}^{\lfloor \frac{l+3}{2} \rfloor} C^j([0, T]; H^{l+2-2j}(I)), \quad \mathbf{v} \in \bigcap_{j=0}^{\lfloor \frac{l+3}{2} \rfloor} W^{j,\infty}(0, T; H^{l+3-2j}(I)),$$

with initial datum $\mathbf{v}_0 \in H^{l+3}(I)$. The uniqueness of the solution is a consequence of a standard energy estimate of the difference of two solutions with the same initial datum. Finally, since problem (1.5) can be solved reverse in time, the continuity with respect to t can be recovered by the same argument given in Kato [6], and this proves Theorem 2.2.

References

- [1] M. Aiki and T. Iguchi, *Motion of a vortex Filament in the half space*, Nonlinear Anal., **75** (2012), pp. 5180–5185.
- [2] V. Banica and L. Vega, *On the Stability of a Singular Vortex Dynamics*, Commun. Math. Phys., **286**(2009), pp. 593–627.
- [3] V. Banica and L. Vega, *Scattering for 1D cubic NLS and singular vortex dynamics*, J. Eur. Math. Soc., **14**(2012), pp. 209–253.

- [4] S. Gutiérrez, J. Rivas, and L. Vega, *Formation of Singularities and Self-Similar Vortex Motion Under the Localized Induction Approximation*, Comm. Partial Differential Equations, **28**(2003), no. 5 and 6, pp. 927–968.
- [5] H. Hasimoto, *A soliton on a vortex filament*, J. Fluid Mech., **51** (1972), no. 3, pp. 477–485.
- [6] T. Kato, *Nonstationary Flows of Viscous and Ideal Fluids in \mathbf{R}^3* , J. Functional Analysis, **9** (1972), pp. 296–305.
- [7] N. Koiso, *The Vortex Filament Equation and a Semilinear Schrödinger Equation in a Hermitian Symmetric Space*, Osaka J. Math., **34** (1997), no. 1, pp. 199–214.
- [8] Nishiyama, *Existence of a solution to the Mixed Problem for a Vortex Filament Equation with an External Flow Term*, J. Math. Scie. Univ. Tokyo, **7** (2000), no. 1, pp. 35–55.
- [9] T. Nishiyama and A. Tani, *Initial and Initial-Boundary Value Problems for a Vortex Filament with or without Axial Flow*, SIAM J. Math. Anal., **27** (1996), no. 4, pp. 1015–1023.
- [10] T. Nishiyama and A. Tani, *Solvability of the localized induction equation for vortex motion*, Comm. Math. Phys., **162** (1994), no. 3, pp. 433–445.
- [11] J. B. Rauch and F. J. Massey, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Math. Soc., **189** (1974), pp. 303–318.
- [12] V. A. Solonnikov, *An initial-boundary value problem for a Stokes system that arises in the study of a problem with a free boundary*, Proc. Steklov Inst. Math., **3** (1991), pp. 191–239.

Masashi Aiki

Department of Mathematics

Faculty of Science and Technology, Tokyo University of Science

2641 Yamazaki, Noda, Chiba 278-8510, Japan

E-mail: aiki_masashi@ma.noda.tus.ac.jp